

# GROUP ALGEBRAIC STUDY OF PERMUTATION - INVARIANT FILTERS

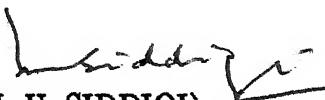
A Thesis Submitted  
in Partial Fulfilment of the Requirements  
for the Degree of  
MASTER OF TECHNOLOGY

*by*  
R. HARI PRASADA RAO

*to the*  
DEPARTMENT OF ELECTRICAL ENGINEERING  
INDIAN INSTITUTE OF TECHNOLOGY KANPUR  
MAY 1990

## CERTIFICATE

Certified that this work 'GROUP ALGEBRAIC STUDY OF PERMUTATION-INVARIANT FILTERS' by Mr. R. Hari Prasada Rao has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

  
(M. U. SIDDIQI) 14-5-80  
Professor

Department of Electrical Engineering  
Indian Institute of Technology  
Kanpur.

EE-1990-M-RAO-GRO

- 7 FEB 1991

CENTRAL LIBRARY  
U. S. AIR FORCE

Acc. No. A.110029

TH

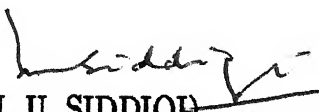
621.3815

R 189

990029

# CERTIFICATE

Certified that this work 'GROUP ALGEBRAIC STUDY OF PERMUTATION-INVARIANT FILTERS' by Mr. R. Hari Prasada Rao has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

  
(M. U. SIDDIQI) 14-5-80

Professor

Department of Electrical Engineering  
Indian Institute of Technology  
Kanpur.



## ABSTRACT

Group algebraic approach to the study of P-I filters is considered. Classification of P-I filters based on the ideals in group algebra is proposed and the possibility of equivalent realization of cyclic permutation invariant (P-I) filters in terms of abelian P-I filters is investigated. It is shown that such realizations are possible only for those P-I filters which are invariant relative to both cyclic and abelian permutation groups of same degree. Equivalent realizations are obtained on the basis of Mixed-Radix (MRX) mapping and mapping based on Chinese-Remainder Theorem (CRT) for integers. Complete characterization of such filters is given for both of these mappings. Implementational advantages of realizing cyclic P-I filters as abelian P-I filters from the point of view of structural concurrency are pointed out.

In the group algebra representation of a signal space, the space consisting of all  $n$ -length real-valued sequences, a P-I filter is a mapping from group algebra into an ideal of the group algebra. A class of P-I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. As a consequence, cyclic P-I filters are classified based on the ideals in cyclic group algebra and abelian P-I filters are classified based on the ideals in abelian group algebra. An expression for the number of conjugacy classes in abelian group algebra is given and its significance to the above classification is pointed out. This classification is instrumental in the identification of filters that are invariant under both cyclic and abelian permutations. Such filters are interpreted as elements of cyclically closed ideals in abelian group algebras and are called *cyclically closed abelian P-I filters*. MRX mapping provides a basis to the characterization of such ideals. From these ideals, a complete characterization of cyclically closed abelian P-I filters is obtained.

Whenever the orders of component cyclic sub groups are pairwise relatively prime integers, mapping based on CRT for integers is considered and it is shown that any cyclic P–I filter, whose dimension is a product of pairwise relatively prime integers has, an equivalent abelian P–I filter realization. It is shown that the same sample domain CRT mapping relates the respective transform domain coefficients. Therefore, in the transform domain, this equivalent abelian P–I filter realization has an implementation advantage over traditional cyclic P–I filter realization using Good–Thomas FFT algorithm in respect of address shuffling.

## ACKNOWLEDGEMENTS

I am grateful to my thesis supervisor, Dr. M. U. Siddiqi, for suggesting the topic for study. His constant encouragement, understanding and constructive criticism helped a long way in shaping the thesis to the present form.

The stimulating discussions with Madhu and Uday are gratefully acknowledged.

I thank Ramprasad for his invaluable help at critical times. I thank Verghese and Ranade for their help in the preparation of this thesis. Finally, I would thank Venu, Hussain, Rajan and all other friends for their company.

**To**  
**My Parents and Teachers**

# TABLE OF CONTENTS

	Page
LIST OF SYMBOLS	ix
LIST OF TABLES	xi
LIST OF FIGURES	xiii
CHAPTER 1 INTRODUCTION	1
1.1 Motivation	1
1.2 Scope of the work	2
1.3 Proposed Line of Approach	4
1.4 Outline of Chapters	6
CHAPTER 2 CLASSIFICATION OF P-I FILTERS BASED ON IDEALS IN GROUP ALGEBRA	7
2.1 P-I Filters	8
2.2 Review of Group Algebraic Concepts	13
2.3 Interpretation of a Class of P-I Filters as an Ideal in Group Algebra	17
2.4 Brief Review of Transform Domain Characterization of Ideals in Group Algebra	20

2.5 Transform Domain Classification of P-I Filters	28
---	----

## CHAPTER 3 CHARACTERIZATION OF CYCLICALLY CLOSED

### ABELIAN P-I FILTERS 31

3.1 Cyclically Closed Abelian P-I Filters	33
3.2 Cyclically Closed Abelian P-I Filters as Elements of Cyclically Closed Abelian Ideals	37
3.3 Identification of Cyclically Closed Ideals in Abelian Group algebra	39
3.4 Characterization of Cyclically Closed Abelian P-I Filters	51
3.5 Transform Domain Characterization of Cyclically Closed Abelian P-I Filters	55
3.6 A Procedure for Identifying 2-D Cyclically Closed Abelian P-I Filters	62
3.7 Equivalent Cyclic P-I Filters based on CRT Mapping	67

## CHAPTER 4 CONCLUSIONS 74

4.1 Summary of the Results	74
4.2 Suggestions for Further Research	75

## REFERENCES 77

## LIST OF SYMBOLS

- $Z_n$  : Set of integers modulo  $n$   
 $\mathbb{R}$  : Field of real numbers  
 $\mathbb{R}^n$  :  $n$ -dimensional vector space over  $\mathbb{R}$   
 $a$  : an  $n$ -tuple belonging to  $\mathbb{R}^n$   
 $M$  : Mixed-Radix set  $\{ m_0, m_1, \dots, m_{r-1} \}$  of  $n$   
 $\langle i_0 i_1 \dots i_{r-1} \rangle$  : Mixed-Radix representation of  $i$  w.r.t.  $M$   
 $\ominus$  : Subtraction in Mixed-Radix number system  
 $PG$  : Transitive abelian permutation group of degree  $n$   
 $PC$  : Transitive cyclic permutation group of degree  $n$   
 $C_{m_i}$  : Cyclic group of order  $m_i$   
 $x_i$  : Generator of  $C_{m_i}$   
 $\otimes$  : Direct product  
 $G$  : Abelian group,  $(= C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}})$ , of order  $n$   
 $X$  : General element of  $G$   
 $X^i$  :  $x_0^{i_0} \cdot x_1^{i_1} \dots x_{r-1}^{i_{r-1}}$   
 $RG$  : Group algebra of  $G$  over  $\mathbb{R}$   
 $a(X)$  : General element of  $RG$   
 $Z[a(X)]$  : Cyclic shift of  $a(X)$   
 $X^i[a(X)]$  : Abelian permutation of  $a(X)$

$\theta(x_i)$  : Minimal idempotent of  $C_{m_i}$  algebra.

$R_0(x_i)$  : Formal sum consisting of all the elements in  $C_{m_i}$

$\theta(X)$  : Idempotent generator of  $RG$

$\langle \theta(x_i) \rangle$  : Ideal generated by  $\theta(x_i)$

$C_n$  : Cyclic group of order  $n$

$Z$  : Generator of  $C_n$

$RC$  : Cyclic group algebra over  $\mathbb{R}$

$a(Z)$  : General element of  $RC$

$\oplus$  : Direct sum

$\mathbb{C}$  : Complex space

$\mathbb{C}_r^n$  : Complex space isomorphic to  $\mathbb{R}^n$

$CG_r$  : Group algebra over  $\mathbb{C}_r^n$

$t_i$  : Number of conjugacy classes in  $C_{m_i}$  algebra

$E^j(X_i)$  : Minimal idempotent generator of  $C_{m_i}$  algebra over a complex space  
restricted to the real images

$E^j(X)$  :  $J^{\text{th}}$  minimal idempotent of  $CG_r$

$\langle E^j(X_i) \rangle$  : Ideal generated by  $E^j(X_i)$

$CC_r$  : Cyclic group algebra over  $\mathbb{C}_r^n$

$E^j(Z)$  : Minimal idempotent generator of  $CC_r$

$\langle E^j(Z) \rangle$  : Ideal generated by  $E^j(Z)$

□ End of proof



# LIST OF TABLES

Table No.	Title	Page
2.4.1	List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_8$ Group Algebra	25
2.4.2	List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_4 \otimes C_2$ Group Algebra	25
2.4.3	List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_{32}$ Group Algebra.	26
2.4.4	List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_4 \otimes C_8$ Group Algebra.	27
3.3.1	List of Minimally Cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_2$ Group Algebra	49
3.3.2	List of Minimally Cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_8$ Group Algebra	49

3.3.3	List of Minimally Cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_4 \otimes C_2$ Group Algebra	50
3.5.1	List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P-I Filters in $C_4 \otimes C_2$ Group Algebra	58
3.5.2	List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P-I Filters in $C_4 \otimes C_8$ Group Algebra	58
3.5.3	List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P-I Filters in $C_4 \otimes C_4 \otimes C_2$ Group Algebra	59

## LIST OF FIGURES

Fig No:	Title	Page
2.1.1	Illustration of the Notions of P-I System P-I Filter and Minimal P-I Filter	11
3.1.1	Sample Domain Interpretations of Cyclically Closed Abelian P-I filter	36
3.5.1	Transform Domain Interpretations of Cyclically Closed Abelian P-I Filter	57
3.7.1	Transform Domain Implementations of a Cyclic P-I Filter using G-T FFT Algorithm and using GWHT based on CRT mapping	73

# CHAPTER 1

## INTRODUCTION

### 1.1 Motivation

Signals are defined as real-valued functions of a variable that takes values from an infinite index set. This index set has the structure of an infinite abelian group. A linear time-invariant (LTI) system is a linear convolution operator that convolves an input signal with its impulse response signal thereby producing an output signal. Fourier transform (FT) provides a basis for the transform domain study of LTI systems. Filtering of a signal by an LTI system means modification of the fourier spectrum of the input signal by an appropriately chosen LTI system. To process finite length signals using LTI systems, zeros are appended to the signals and these are treated as signals defined over infinite index set with a finite support. Alternatively index set can be treated as finite to deal with finite length signals. When the structure of index set is a finite cyclic group, we have a known class of cyclic convolution systems (CCSs). Here, discrete fourier transform (DFT) plays a role similar to that of FT for the case of LTI systems. The basis functions for DFT are discrete complex exponentials and are ordered in terms of frequency of discrete complex exponentials. Hence the notion of filtering with respect to frequency as low pass, high pass, band pass and band stop is easily introduced. Another well-known class of dyadic convolution systems (DCSs) are obtained by assuming finite dyadic group structure to the finite index set. Walsh-Hadamard transform (WHT) provides the necessary basis for the transform domain study of this class of systems. Here also the discrete Walsh functions which are the basis functions for WHT are ordered in terms of number of zero crossings generally known as sequency. Therefore, analogous to the notion of filtering with respect to frequency using CCSs, we have the notion of filtering using DCSs with respect to sequency.

Generalization of the structure of finite index set to a finite abelian group results in a more general class of convolution systems known as permutation invariant (P-I) convolution systems. In such systems, generalized Walsh-Hadamard transforms (GWHTs) play a role similar to that of DFT and WHT respectively in the context of CCSs and DCSs. In fact, CCSs and DCSs are special cases of the general class of P-I convolution systems. Filtering of signals using P-I convolution system that does not belong to either cyclic class or dyadic class, means modifying the GWHT spectrum of input signals by an appropriate P-I convolution system. In this general case, the criterion of number of zero crossings may be used to find an ordering of the basis functions for GWHT as does the notions of frequency and sequency in the case of CCS and DCS P-I systems respectively. It remains to be seen whether this provides a viable alternative. It is obvious that for such GWHT, the number of zero crossings of basis functions will be lower bounded by the zero crossings (frequency) of a corresponding DFT basis functions and upper bounded by the zero crossings (sequency) of the corresponding basis functions of WHT (whenever it is applicable). In this context, it is worth while to compare the effects of filtering by different classes of P-I filters. Towards this end, experimentally, given an image, some of the conjugate positions were forced to zero, and the effect of this type of filtering was compared both with respect to cyclic and non-cyclic class of P-I systems of the same order. It was noticed that upon forcing certain conjugacy classes to zero produced same output images in both cases. This observation motivated to study whether there are any systems that belong to both cyclic and abelian classes and the possibility of equivalent realization of cyclic permutation invariant (P-I) systems in terms of abelian P-I systems.

## 1.2 Scope of the Work

This thesis is concerned with the study of equivalent realization of cyclic P-I filters in terms of abelian P-I filters. For a given class of abelian P-I filters, we obtained a complete characterization of P-I filters that are invariant under both cyclic and abelian

permutations. For this characterization, MRX mapping provides the necessary basis. However, when the orders of the component cyclic subgroups are pairwise relatively prime integers, mapping based on CRT is also possible and it is shown that all cyclic P-I filters whose dimension are products of integers which are pairwise relatively prime have an equivalent abelian P-I filter realizations under this mapping.

Signal space, the space consisting of all  $n$ -length real valued sequences, can be viewed either as a vector space,  $\mathbb{R}^n$ , or as a group algebra,  $\mathbb{R}G$ , where  $G$  is an abelian group of order  $n$ . A finite discrete system is a linear transformation,  $T$ , which is a mapping from  $\mathbb{R}^n$  into  $\mathbb{R}^n$ . A system is said to be permutation invariant (P-I), if the input signal is permuted in a particular manner then, the output signal of the system is also permuted in the same manner. On given number of  $n$ -samples of an input signal, there exists  $n!$  permutations. Each class of P-I systems of dimension  $n$  is defined with respect to a set of permutations, forming transitive abelian permutation (TAP) group,  $PG$ , of degree  $n$ . The number of distinct classes of P-I systems of dimension  $n$  is equal to the number of distinct TAP groups of degree  $n$ , in the  $n!$  permutations. A P-I system,  $T$ , defined invariant relative to a TAP group,  $PG$ , of degree  $n$  which is isomorphic to  $G$ , is called an abelian P-I filter if it maps any  $n$ -tuple of  $\mathbb{R}^n$  into an  $n$ -tuple of a linear subspace of  $\mathbb{R}^n$  which is closed under the permutations belonging to  $PG$ . Input signal space to an abelian P-I filter is the whole  $\mathbb{R}^n$  and output signal space is a linear subspace of  $\mathbb{R}^n$  closed under permutations of  $PG$ . A cyclic P-I filter is a P-I system,  $T$ , defined invariant relative to a cyclic TAP group,  $PC$ , of degree  $n$ , whose output signal space is a linear subspace of  $\mathbb{R}^n$  closed under cyclic permutations belonging to  $PC$ . Linear P-I filters have matrix representations and these matrices are called P-I matrices. The zero<sup>th</sup> column of a P-I matrix is the unit-sample response of the filter and other columns of this matrix are just the permutations (of the TAP group w.r.t. which the filter is defined) of the zero<sup>th</sup> column. Any output signal of a P-I filter is a linear combination of the column vectors of the P-I matrix with the samples of the input signal as weighting factors.

Suppose, if on a particular vector of  $\mathbb{R}^n$ , the effect of an abelian permutation of PG is same as that of a cyclic permutation of PC, then a P-I filter whose P-I matrix has got this type of vector as its zero<sup>th</sup> column maps any n-tuple of  $\mathbb{R}^n$  into an n-tuple of a linear subspace of  $\mathbb{R}^n$  closed under both abelian as well as cyclic permutations belonging to PG and PC respectively. Such a P-I filter is called a cyclically closed abelian P-I filter. The main objective of this thesis is the characterization of such P-I filters based on both MRX and CRT mapping. For characterization purposes it will be advantageous to work in the group algebraic framework.

### 1.3 Proposed Line of Approach

There are three basic steps involved towards characterizing cyclically closed abelian P-I filters. The first step is to decompose  $\mathbb{R}^n$  into a direct sum of smallest subspaces closed under the permutations belonging to PG. The second step is to identify which of these smallest subspaces or direct sum of subsets of these smallest subspaces, are closed under cyclic permutations belonging to PC. The last step is to identify the vectors whose cyclic permutations are same as abelian permutations from these spaces. All these steps are carried out by making use of the properties of group algebraic structure of signal space. Then P-I filtering is nothing but multiplication of group algebraic representations of input and filter vectors. Group algebra,  $RG$ , of a group  $G$  which is isomorphic to PG, over the field of real numbers can be expressed as a direct sum of minimal ideals in that group algebra. Any other ideal is a direct sum of subset of these minimal ideals. Every minimal ideal has an unique idempotent generator. This generates all the elements of the corresponding minimal ideal. Characterization of these idempotent generators is done easily in the transform domain. GWHT is an isomorphism between  $RG$  and the complex space restricted to the real images. In GWHT domain each of the minimal ideals in  $RG$  corresponds to a set of elements which are non-zero over a single common conjugacy class. By invoking some properties of minimal ideals, it is shown that minimal ideals are

isomorphic to smallest subspaces closed under the permutations belonging to PG and that there is one to one correspondence between the ideals in group algebra and the subspaces of  $\mathbb{R}^n$  closed under the permutations belonging to PG. This fact leads to the interpretation that abelian P-I filter is a mapping from RG to an ideal and a set of filters whose transform vectors are non-zero over a single common conjugacy class forms a minimal ideal. Consequently, cyclic P-I filters are classified based on the ideals in cyclic group algebra and abelian P-I filters are classified based on the ideals in abelian group algebra. This classification is made use of in the characterization of cyclically closed abelian P-I filters. For a given class of abelian P-I filters, we characterize all the elements of abelian group algebra whose cyclic permutations are same as the corresponding abelian permutations. This correspondence is established on the basis of mixed-radix (MRX) mapping. Because of the necessity that these elements should belong to the ideals of abelian group algebra that are closed under cyclic permutations, firstly, we completely characterize all the ideals that are closed under cyclic permutations in the abelian group algebra. This characterization is given in [1] for the group algebra over a finite field. It is noticed that this result is also applicable to the case of Real field and is used for characterizing cyclically closed abelian ideals in RG. Since, not all the elements of cyclically closed ideals in abelian group algebra qualify as cyclically closed abelian P-I filters, we characterize all those elements which are cyclically closed abelian P-I filters. Further, we consider CRT mapping (When the orders of component cyclic groups are pair wise relatively prime) and show that under this mapping, a cyclic group algebra is isomorphic to an abelian group algebra. Then we show that all cyclic P-I filters in such a cyclic group algebra have an equivalent realization as an abelian P-I filters. A brief summary of results obtained is given below.

- 1) P-I filters are classified based on the ideals in group algebra.
- 2) A complete characterization of cyclically closed abelian P-I filters is given based on MRX mapping.



- 3) A procedure for identification of 2-D cyclically closed abelian P-I filters is given.
- 4) It is shown that any cyclic P-I filter whose dimension is a product of pairwise relatively prime integers can be realized as an equivalent abelian P-I filter.

## 1.4 Outline of Chapters

In Chapter 2, P-I filters are classified based on the ideals in group algebra. This chapter starts with the basic definitions of P-I filters. In Section 2.2, some of the important properties of group algebra are discussed and it is shown that a P-I filter's output signal space is an ideal in a group algebra. In the next section, the set of all P-I filters that maps elements from  $RG$  into an ideal, is shown to be that ideal itself. This provides a basis for classifying P-I filters. Consequently, cyclic and abelian P-I filters are classified based on ideals in their respective group algebras. In Section 2.4, transform-domain characterization of ideals in a group algebra is reviewed. An expression giving the number of conjugacy classes is derived and its significance with respect to above mentioned classification of P-I filters is pointed out. In the last section of this chapter, transform-domain classification of P-I filters is discussed.

Chapter 3 is devoted to the characterization of cyclically closed abelian P-I filters. In Section 3.1, the concept of cyclically closed abelian P-I filter is explained. In the next section cyclically closed P-I filters are shown to be elements of cyclically closed ideals in abelian group algebra. Section 3.3, contains the characterization of cyclic ideals in an abelian group algebra. In the next section, the characterization of cyclically closed abelian P-I filters is given. In Section 3.5, transform-domain characterization of cyclically closed abelian P-I filters is given. In Section 3.6, 2-D P-I filters are discussed and a procedure for identifying 2-D cyclically closed abelian P-I filters is given. In the last section, we discussed equivalent abelian P-I filter realizations of cyclic P-I filters based on CRT mapping.

The thesis is concluded in Chapter 4 giving summary of results obtained and suggestions for further research.

## CHAPTER 2

### CLASSIFICATION OF P-I FILTERS BASED ON IDEALS IN GROUP ALGEBRA

A signal space consisting of all real-valued sequences of length  $n$ , can be treated either as a vector space over reals,  $\mathbb{R}^n$ , or as a group algebra over the real field,  $\mathbb{R}G$  [4]. Usually, a signal space is treated as  $\mathbb{R}^n$  so that a Permutation-invariant (P-I) system turns out to be a linear transformation on  $\mathbb{R}^n$ ,  $T$ , which is a mapping of  $\mathbb{R}^n$  into a P-I subspace of  $\mathbb{R}^n$  [3]. On the other hand, when a signal space is treated as a group algebra,  $\mathbb{R}G$ , a P-I system can be viewed as multiplication of elements belonging to the group algebra  $\mathbb{R}G$ , with the unit sample response element  $\in \mathbb{R}G$ , characterizing the P-I system.

In this chapter, we consider group algebra representation of a signal space and consequently, classify P-I filters based on the ideals in group algebra.

In Section 2.1, we provide certain preliminary definitions that are required for our study. In Section 2.2, we consider group-algebra representation of a signal space and list out some of the important properties of group-algebra that are relevant for our study. In addition, we show that there is an isomorphic correspondence between the ideals of a group algebra and the P-I subspaces of the vector space  $\mathbb{R}^n$ . In Section 2.3, a class of P-I filters that maps elements of a group algebra into an ideal is shown to be that ideal itself. In Section 2.4, transform-domain characterization of ideals in  $\mathbb{R}G$  is reviewed and an expression for the number of conjugacy classes is derived. Finally, in Section 2.5, we present the transform-domain classification of P-I filters.

## 2.1 Permutation-Invariant (P-I) Filters

In this section treating the signal space, the space consisting of all  $n$ -length real-valued sequences, as an  $n$ -dimensional vector space over the field of real numbers,  $\mathbb{R}^n$ , the concept of P-I filter is being explained.

Let us consider the following three  $n$ -tuples,  $a = (a_0 \ a_1 \ a_2 \ \dots \ a_j \ \dots \ a_{n-1})$ ,  $b = (b_0 \ b_1 \ b_2 \ \dots \ b_{j-1} \ b_{n-1})$  and  $s = (s_0 \ s_1 \ s_2 \ \dots \ s_j \ \dots \ s_{n-1})$  from  $\mathbb{R}^n$  denoting the input output and unit-sample response sequences of a system respectively.

Let PG be the transitive abelian permutation group of degree  $n$ . The effect of any permutation,  $p_k$ , belonging to PG on any signal,  $a = (a_0 \ a_1 \ a_2 \ \dots \ a_j \ \dots \ a_{n-1}) \in \mathbb{R}^n$ , is described by the following relationship.

$$p_k(a) = \langle a_{0\theta k} \ a_{1\theta k} \ \dots \ a_{j\theta k} \ \dots \ a_{n-1\theta k} \rangle ; k \in Z_n \quad (2.1.1)$$

$$\text{where } j\theta k = \langle (j_0 - k_0)_{m_0} \ (j_1 - k_1)_{m_1} \ \dots \ (j_{r-1} - k_{r-1})_{m_{r-1}} \rangle \quad (2.1.2)$$

A system,  $T$ , which is defined with respect to the transitive abelian permutation group, PG, of degree  $n$ , whose elements,  $p_k$ ,  $k \in Z_n$ , are being specified by the mixed-radix set  $\{m_0 \ m_1 \ \dots \ m_{r-1}\}$ , is characterized by the following generalized convolutional relationship.

$$b_i = \sum_{j=0}^{n-1} s_{i\theta j} a_j \quad (2.1.3)$$

$$\text{where } i\theta j = \langle (i_0 - j_0)_{m_0} \ (i_1 - j_1)_{m_1} \ \dots \ (i_{r-1} - j_{r-1})_{m_{r-1}} \rangle$$

Alternatively (2.1.3) can also be expressed in the form of matrix-equation as

$$b = Sa ; a \text{ and } b \text{ are } n \times 1 \text{ matrices and } S \text{ is } n \times n \text{ matrix} \quad (2.1.4)$$

Elements of  $S$  are given by

$$S_{i,j} = s_{i\theta j} \quad (2.1.5)$$

where  $i\theta j$  is as defined above.

**Example 2.1.1:** For the class of P-I systems of dimension  $n$  defined with respect to the cyclic permutation group PG of degree  $n$ , the generalized convolutional relationship takes the familiar form of the cyclic convolution relationship

$$b_i = \sum_{j=0}^{n-1} s_{(i-j) \bmod n} a_j \quad ; \quad i \in Z_n \quad (2.1.6)$$

representing cyclic P-I systems commonly known as cyclic convolution systems.

Hereafter, we refer to the above defined system as system,  $T$ , invariant relative to PG. It is understood that PG is being defined with respect to the mixed-radix set  $\{m_0, m_1, \dots, m_{r-1}\}$ .

The matrix,  $S$ , in (2.1.5) is also known as a permutation-invariant matrix, and its generator, i.e., the zero<sup>th</sup> column,  $s = (s_0, s_1, s_2, \dots, s_j, \dots, s_{n-1})$  is actually the unit-sample response of the system,  $T$ . Any other column,  $k$ , of  $S$  can be obtained by permuting the zero<sup>th</sup> column,  $s$ , with the  $k^{\text{th}}$  member,  $p_k$ , of PG, in accordance with (2.1.1). This fact leads to the interpretation that, the output sequence,  $b$ , of  $T$  can be interpreted as a linear combination of its column vectors with the samples of the input sequence,  $a$ , as weighting factors. In the form of equation, it is represented as

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \end{bmatrix} = \sum_{k=0}^{n-1} a_k \begin{bmatrix} s_{0\theta k} \\ s_{1\theta k} \\ \vdots \\ s_{j\theta k} \\ \vdots \\ s_{n-1\theta k} \end{bmatrix} \quad (2.1.7)$$

In other words, the system,  $T$ , defined with respect to PG, maps the space of real  $n$ -tuples,  $\mathbb{R}^n$ , into the space of real  $n$ -tuples, in which each of the  $n$ -tuples is a linear combination of the column vectors of  $S$ . It means, the output signal space of the system,  $T$ ,

has the following sequences:

- a) Zero<sup>th</sup> column,  $s$ , of  $S$
- b) All the resulting sequences, by the application of permutations belonging to  $PG$  to  $s$ .
- c) The sequences obtained by the linear combinations of the sequences mentioned in (a) and (b).

This output signal space of  $T$  being characterized by  $s$  is linear and is closed under the permutations belonging to  $PG$ . In the following discussion, this type of space is simply referred to as the linear space invariant relative to  $PG$ . When the whole  $\mathbb{R}^n$  is considered, it is obviously a linear space invariant relative to  $PG$  and therefore a valid output signal space for  $T$ . From the space of real  $n$ -tuples,  $\mathbb{R}^n$ , it is possible to find linear subspaces of  $\mathbb{R}^n$  invariant relative to  $PG$ . These subspaces also can be taken as valid output signal spaces for the system,  $T$ , as defined in (2.1.3). In this context, the system,  $T$ , which is a mapping from  $\mathbb{R}^n$  into a linear subspace of  $\mathbb{R}^n$ , invariant relative to  $PG$ , is called a  $P$ - $I$  filter. In other words,  $P$ - $I$  filter maps any real  $n$ -tuple belonging to  $\mathbb{R}^n$  into a  $n$ -tuple belonging to the linear subspace of  $\mathbb{R}^n$  invariant relative to  $PG$ . Summarizing the above discussion we have the following definition for  $P$ - $I$  filter.

✓ **Definition 2.1.1:** A  $P$ - $I$  filter can be defined as a system,  $T$ , which maps the space of real  $n$ -tuples,  $\mathbb{R}^n$ , into the linear subspace of  $\mathbb{R}^n$ , invariant relative to  $PG$  and its output-input relationship is characterized by (2.1.3).

✓ This  $P$ - $I$  filter is said to be a *minimal  $P$ - $I$  filter* if its output signal space does not contain any proper subspaces invariant relative to  $PG$ . Figure 2.1.1 illustrates the above notions of  $P$ - $I$  system,  $P$ - $I$  filter and minimal  $P$ - $I$  filter.

✓ If the permutation group  $PG$  is cyclic (denote it by  $PC$ ), then a  $P$ - $I$  filter is called a *cyclic  $P$ - $I$  filter*. Otherwise, it is called as an *abelian  $P$ - $I$  filter*. It is to be noted that the output signal space of a cyclic  $P$ - $I$  filter is a linear subspace of  $\mathbb{R}^n$  closed under the permutations of  $PC$ . On the other hand, output signal space of an abelian  $P$ - $I$  filter is a

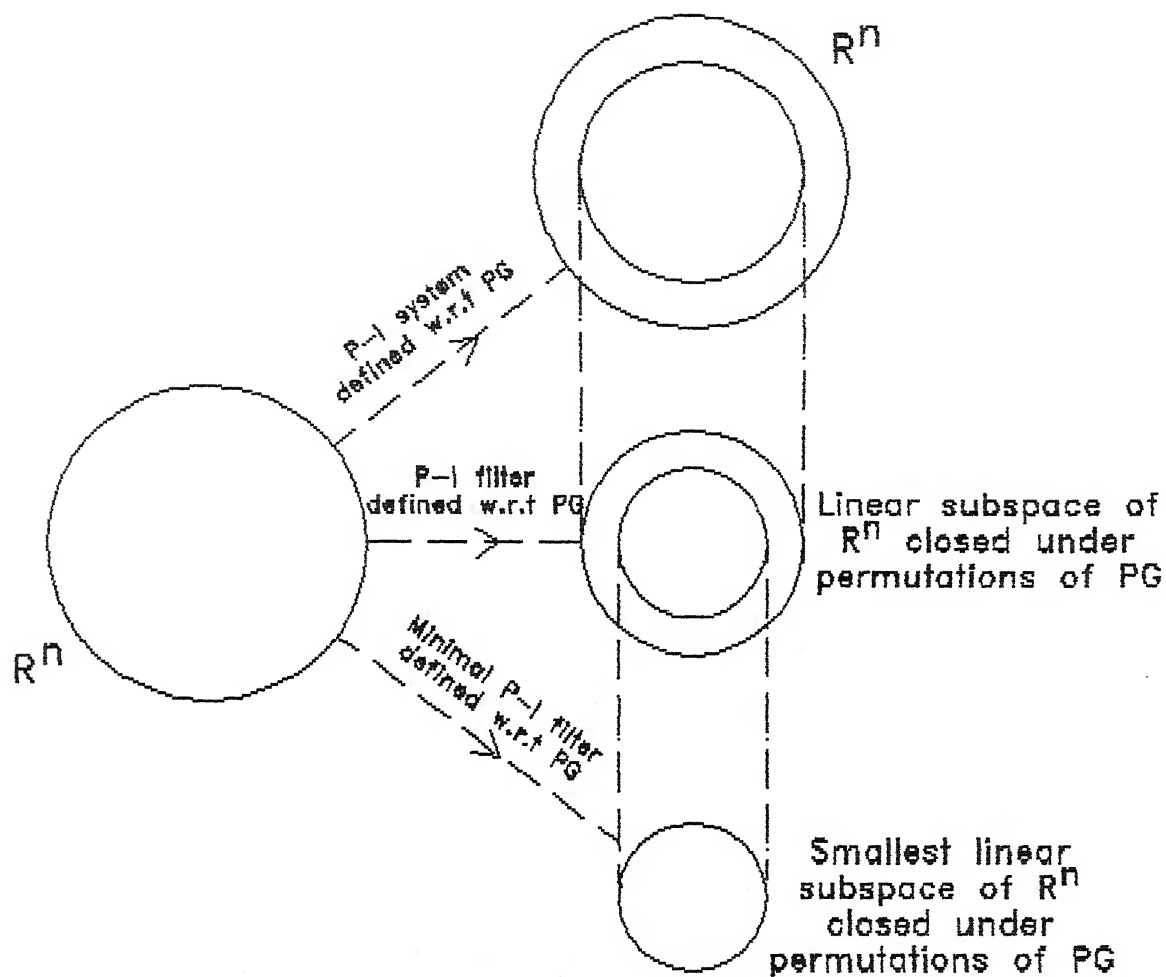


Fig 2.1.1 Illustration of the notions of P-I System, P-I Filter and Minimal P-I Filter.

## 2.2 Review of Group Algebraic Concepts

In this section the signal space is being treated as a group-algebra [4] and then isomorphic correspondence is established between the ideals in group algebra and the permutation invariant subspaces in  $\mathbb{R}^n$ . This section starts with indexing the elements of an abelian group using mixed-radix indexing scheme.

The elements of an abelian group,  $G$ , of order  $n$  which is a direct product of cyclic subgroups with generators  $x_0, x_1, \dots, x_{r-1}$  of orders  $m_0, m_1, \dots, m_{r-1}$  respectively, are indexed as follows.

$$G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}} \dots ; n = m_0 m_1 \dots m_{r-1}$$

where  $C_{m_i}$  is cyclic subgroup of order  $m_i$  with generator  $x_i$  and  $\otimes$  denotes direct product.

$$X^0 = 1$$

$$X^1 = x_0$$

$$X^2 = x_0^2$$

$$X^{m_0-1} = x_0^{m_0-1}$$

$$X^{m_0} = x_1$$

$$X^{m_0+1} = x_1 x_0$$

$$X^i = x_{r-1}^{i_{r-1}} x_{r-2}^{i_{r-2}} \dots x_1^{i_1} x_0^{i_0} = \prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}} \quad (2.2.1)$$

where  $i_{\alpha}$  is the index of  $i$  with respect to the mixed-radix  $m_{\alpha}$  in the mixed-radix representation of  $i$  with respect to the mixed-radix set  $\{m_0, m_1, \dots, m_{r-1}\}$

$$X^{n-1} = x_{r-1}^{m_{r-1}-1} x_{r-2}^{m_{r-2}-1} \dots x_1^{m_1-1} x_0^{m_0-1}$$

The  $i^{\text{th}}$  element  $X^i$  of the abelian group,  $G$ , is indexed using mixed-radix indexing

scheme as given in (2.2.1). In the following discussion  $X$  is used to denote a general element belonging to  $G$ .

Any  $n$ -tuple,  $a = (a_0 \ a_1 \ a_2 \ \dots \ a_j \ \dots \ a_{n-1})$  belonging to  $\mathbb{R}^n$  can be expressed as a polynomial in variables  $x_0, x_1, \dots, x_{r-1}$  as follows.

$$a(x_0, x_1, \dots, x_{r-1}) = \sum_{i=0}^{n-1} a_i \left( \prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}} \right) \quad (2.2.2)$$

where  $i_{\alpha}$  is as defined in (2.2.1)

Let us denote the term in brackets of (2.2.2) by  $X^i = \prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}}$ . In the light of above discussion on indexing the elements of abelian group,  $G$ , of order  $n$  it is to be noted that the set of  $X^i$ 's,  $\{X^0, X^1, \dots, X^{n-1}\}$ , has the structure of an abelian group of order  $n$ .

The polynomial  $a(x_0, x_1, \dots, x_{r-1})$  can be interpreted as formal sums of the form  $\sum_{i=0}^{n-1} a_i \left( \prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}} \right)$  where  $a_i \in \mathbb{R}$  and  $x_{\alpha}^{i_{\alpha}} \in C_{m_{\alpha}}$ , a cyclic group of order  $m_{\alpha}$  with generator  $x_{\alpha}$ . This set of formal sums have an algebraic structure called group algebra.

**Definition 2.2.1:** Let  $G$  be an abelian group of order  $n$  and  $\mathbb{R}$  real field. The set of formal sums denoted by  $RG$  where  $RG = \{a(X) = \sum_{X \in G} a_X X ; a_X \in \mathbb{R}\}$ , with the two operations addition and multiplication respectively as

$$a(X) + s(X) = \sum_{X \in G} (a_X + s_X) X ; a(X) = \sum_{X \in G} a_X X , s(X) = \sum_{X \in G} s_X X \quad (2.2.3)$$

and for  $Y \in G$

$$Ya(X) = \sum_{X \in G} a_X XY = \sum_{X \in G} a_{XY^{-1}} X \quad (2.2.4)$$

is called group algebra.

**Remark 2.2.1:** Combining the operations given by (2.2.3) and (2.2.4) we get, for  $A, B \in RG$

$$a(X)s(X) = \left( \sum_{X \in G} a_X X \right) \left( \sum_{X \in G} s_X X \right) = \sum_{W \in G} \left( \sum_{XY=W} a_X s_Y \right) W \quad (2.2.5)$$

The term inside the brackets of last expression of (2.2.5) is a convolution



determined by the structure of the abelian group  $G$ . This convolution has another interpretation as multiplication of polynomials modulo polynomial  $(x_0^{m_0} - 1), (x_1^{m_1} - 1), \dots, (x_{r-1}^{m_{r-1}} - 1)$  when  $i^{\text{th}}$  element,  $X^i$ , of  $G$  is indexed as in (2.2.1).

As a consequence, cyclic convolution defined by (2.1.6) and generalized convolution defined by (2.1.3) can be interpreted as

$$b(Z) = a(Z)s(Z) \bmod (Z^n - 1) \quad (2.2.6)$$

where  $Z$  is the generator of cyclic group  $C$  of order  $n$ , and

$$b(X) = a(X)s(X) \bmod (x_0^{m_0} - 1), (x_1^{m_1} - 1), \dots, (x_{r-1}^{m_{r-1}} - 1) \quad (2.2.7)$$

respectively.

**Remark 2.2.2:** From (2.2.3) we see that group algebra,  $RG$ , has the structure of a vector space over  $\mathbb{R}$  of dimension  $n$  equal to the order of the group  $G$ , denoted by  $\mathbb{R}^n$ . Group algebra can be considered as a ring with addition and multiplication defined as in (2.2.3) and (2.2.4) respectively. Moreover it is a commutative ring with identity.

We now proceed to discuss the notion of Ideals in group algebra.

**Definition 2.2.2:** A subset  $I$  of group algebra,  $RG$ , is called an Ideal if

- a)  $I$  is a linear subspace of  $\mathbb{R}^n$
- b) For all  $a(X) \in I$  and  $X \in G$ ,  $Xa(X)$  should be in  $I$ .

In the following discussion, some of the results from semi-simple group algebra which are having relevance to our context will be presented [7].

The group algebra  $RG$  over the field of real-numbers is semi-simple group algebra. From Remark (2.2.2) group algebra  $RG$  can be viewed as a ring (commutative ring with identity). Semi-simple group algebra  $RG$  is a principal ideal ring ; that is every ideal,  $I$ , is of the form

$$I = \{s(X)a(X), s(X) \in RG\} \text{ for some element } a(X) \in RG.$$

An ideal  $I$  generated by  $s(X)$  is denoted by  $\langle s(X) \rangle$ . In semi-simple group algebra,

✓ every ideal has an idempotent generator  $\theta(X)$ . This idempotent generator has the following properties.

$$\text{Let } \theta(X) = \sum_{i=0}^{n-1} \theta_X X^i \quad (2.2.8)$$

$$\text{✓ } [\theta(X)]^2 = \theta(X)$$

$$\text{✓ } s(X) \in I \iff s(X)\theta(X) = s(X) \quad (2.2.9)$$

It is also known from the theory of semi-simple group algebras that  $RG$  is the direct sum of its minimal ideals. A minimal ideal is one which has no proper sub-ideals.

$$\text{✓ } RG = \langle M_0 \rangle \oplus \langle M_1 \rangle \oplus \dots \oplus \langle M_{t-1} \rangle \quad (2.2.10)$$

Any other ideal in  $RG$  can be expressed as a direct sum of subset of these minimal ideals.

The idempotent generators  $\theta_i(X)$  of  $M_i$  satisfy the following properties:

$$\text{✓ } \sum_{i=0}^{t-1} \theta_i(X) = 1 \quad (2.2.11)$$

$$\text{✓ } \theta_i(X)\theta_j(X) = 0 \quad ; i \neq j \quad (2.2.12)$$

$$\langle \theta_i(X) \rangle \cap \langle \theta_j(X) \rangle = 0 \quad ; i \neq j \quad (2.2.13)$$

The results from the group theory that any finite group of order  $n$  is isomorphic to a regular permutation group of the same degree and that a transitive abelian permutation group is a regular permutation group, lead us to see the connection between ideal and the output signal space of the system defined by (2.1.3).

In the light of (2.2.4), condition(b) of definition (2.2.2) can also be interpreted as the set of formal sums, formed by the sequences resulting from the application of permutations belonging to  $PG$  to  $a = (a_0 \ a_1 \ a_2 \ \dots \ a_j \ \dots \ a_{n-1})$ , also should belong to  $I$ . Therefore, output signal space of the system defined by (2.1.3) can also be viewed as Ideal in the group algebra  $RG$ . In the following theorem, we prove that there is one to one correspondence between ideals in group algebra and the subspaces invariant relative to  $PG$ .

✓ **Theorem 2.2.1:** There is one-to-one correspondence between subspaces invariant relative to  $PG$  over  $\mathbb{R}$  and the ideals in the group algebra,  $RG$  of abelian group  $G$ .

Proof: Let  $\mathbb{R}_r$  be a linear subspace of  $\mathbb{R}^n$  and  $RG_r$  denote subspace of  $RG$ . Let

$a = (a_0 \ a_1 \dots a_k \dots a_i \dots a_{i+k} \dots a_{n-1})$ , be a vector of length  $n$  belonging to  $\mathbb{R}^n$ . Consider

$a(X) = \sum_{i=0}^{n-1} a_i \left( \prod_{\alpha=0}^{r-1} x_{\alpha}^i \right)$  as the element in group algebra  $RG$  corresponding to  $a$ . It is

obvious that this association is an isomorphism between  $RG$  and  $\mathbb{R}^n$ , considered as vector spaces over  $\mathbb{R}$ . Hence  $\mathbb{R}_r = RG_r$ . An element  $p_k$ ,  $k \in \mathbb{Z}_n$ , permutes  $a$  into

$$a = (a_0 \ a_1 \dots a_k \dots a_i \dots a_{i+k} \dots a_{n-1})$$

$$p_k(a) = (a_{0\theta k} \ a_{1\theta k} \dots a_{0\theta} \ a_{i\theta k} \dots a_i \dots a_{n-1\theta k})$$

The corresponding element of  $RG$  is

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^{i+k} \quad (2.2.14)$$

Since  $X^{i+k} = X^i X^k$ , (2.2.14) can be written as

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^i X^k = X^k(a(X)) ; X^k \in G \quad (2.2.15)$$

If  $\mathbb{R}_r$  is assumed to be invariant relative to  $PG$ , then  $p_k(a) \in \mathbb{R}_r$  whenever  $a \in \mathbb{R}_r$ . Then according to (2.2.15)

$$a(X) \in RG_r, X^i \in G \longrightarrow X^i a(X) \in RG_r$$

and therefore  $RG_r$  is an ideal in  $RG$ .  $\square$

With the correspondence between ideals and P-I subspaces closed under the permutations of  $PG$ , being thus established, we now move on to define P-I filter in terms of ideals in group algebra.

## ✓ 2.3 Interpretation of a Class of P-I Filters as an Ideal in Group Algebra

From the last section it is known that group-algebra can be expressed as a direct sum of minimal ideals and that any ideal is a direct sum of subset of these minimal ideals. Further, there is one-to-one correspondence between the ideals and the permutation-invariant subspaces. By making use of these facts,  $\mathbb{R}^n$  can be decomposed into

✓  $R_i$ s as  $R^n = R_0 \oplus R_1 \oplus \dots \oplus R_{t-1}$ , where  $R_i$  is the smallest subspace, closed under the permutations of PG, corresponding to the minimal ideal  $M_i$  in RG [5]. Any  $n$ -tuple,  $a$ , of  $R^n$  is uniquely expressed as direct sum of  $n$ -tuples,  $a^0, a^1, \dots, a^{t-1}$  belonging to  $R_0, R_1, \dots, R_{t-1}$  respectively. Since the system,  $T$ , is defined with respect to PG the subspaces  $R_i$ s are invariant under  $T$  also. The fact that each  $R_i$  is invariant under  $T$  enables us to view the action of  $T$  as the independent action of transformations  $T_i$  on the subspaces  $R_i$ , where  $T_i$  is the transformation  $T$  restricted to  $R_i$ . Therefore the action of  $T$  on  $a \in R^n$  can be expressed in the following form.

$$✓ Ta = T_0 a^0 \oplus T_1 a^1 \oplus \dots \oplus T_{t-1} a^{t-1} \quad (2.3.1)$$

In other words, this  $T$  can be described as the direct sum of the transformations  $T_0, T_1, \dots, T_{t-1}$ . Our purpose is to study  $T$  by finding invariant direct-sum decompositions in which the transformations  $T_i$  are of elementary nature. Towards this end, each class of these elementary transformations can be interpreted as a minimal ideal in the group algebra for which well-known methods of characterizations are available in the literature. We now give the definition of a P-I filter in group algebra as follows.

✓ **Definition 2.2.3:** P-I filter is a system,  $T$ , which maps any element of group algebra RG into an element of an ideal in the group-algebra and its input-output relationship is given by

$$✓ b(X) = s(X) a(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), \dots, (X_{r-1}^{m_{r-1}} - 1) \quad (2.3.2)$$

where  $a(X)$ ,  $s(X)$  and  $b(X)$  are input, unit-sample response and output elements of the filter.

For the case of cyclic P-I filter this relation takes the form of

$$✓ b(Z) = s(Z) a(Z) \bmod (Z^n - 1) \quad (2.3.3)$$

where  $Z$  is the generator of cyclic group of order  $n$ .

✓ **Remark 2.3.1:** The unit-sample response element  $s(X)$  of  $T$  can be interpreted as the

direct sum of vectors belonging to the minimal ideals in the group-algebra. Output space of  $T$  is the ideal generated by  $s(X)$ .

✓ **Remark 2.3.2:** A set of P-I filters that maps elements of  $RG$  into the same ideal forms an ideal in group algebra. This follows from the definition of the ideal. We call this set as an ideal of P-I filters. This affords a classification of P-I filters based on the ideals in group algebra  $RG$ . Consequently, a class of cyclic P-I filters that maps into the same ideal can be interpreted as an ideal in cyclic group algebra  $RC$  and abelian P-I filters as an ideal in abelian group algebra  $RG$ .

✓ A P-I filter is said to be *minimal P-I filter* if its output signal space does not contain any nontrivial sub ideals or if the ideal generated by  $s(X)$  is a minimal ideal. A class of P-I filters,  $\{s(x)\}$ , that generate a common minimal ideal is a minimal ideal in  $RG$ . Similarly a class of cyclic P-I filters,  $\{s(Z)\}$ , that map into a common minimal ideal is a minimal ideal in cyclic group algebra  $RC$ . To distinguish the ideals in  $RG$  from the ideals in  $RC$ , we call the former as abelian ideals and the latter as cyclic ideals.

✓ **Remark 2.3.3:** Some of the advantages of classification of P-I filters based on the ideals of group algebra are as follows.

1) This is concerned with the implementation of a P-I filter. In the beginning of this section, we have shown that a P-I filter can be thought of as the direct sum of minimal P-I filters. It is known from the work of T.Beth [8] that it is more efficient to work out computations in each summand minimal ideal than in the group algebra itself. This can be made use of in implementing a P-I filter.

2) In order to characterize a P-I filter in terms of a linearity property say ' $t$ ', sometimes it will be advantageous to characterize an ideal corresponding to that P-I filter in terms of ' $t$ '. This will be made use of while studying cyclically closed abelian P-I filters in the next chapter.

In the following section, we present the transform domain characterization of ideals in group algebra which means the characterization of P-I filters.

## 2.4 Brief Review of Transform Domain Characterization of Ideals

From Section 2.2, it is known that every ideal in  $RG$  is characterized by an idempotent generator. Therefore, by characterizing the minimal idempotent generators in  $RG$ , minimal ideals in  $RG$  can be characterized. Characterization of these Idempotent generators is more easy in the transform domain rather than in the sample domain.

Generalized Walsh-Hadamard transform defined with respect to the mixed-radix set  $\{m_0, m_1, \dots, m_{r-1}\}$  maps  $a = (a_0 a_1 a_2 \dots a_j \dots a_{n-1}) \in \mathbb{R}^n$  into a complex vector  $A = (A_0 A_1 A_2 \dots A_i \dots A_{n-1})$  in the transform domain. These two vectors are related by the following equations.

$$A_i = \sum_{j=0}^{n-1} \left( \prod_{\alpha=0}^{r-1} \gamma_{m_\alpha}^{-i_\alpha a_\alpha} \right) a_j ; \quad i \in Z_n \quad \text{and} \quad \gamma_{m_\alpha} \text{ is } m_\alpha^{\text{th}} \text{ root of unity.} \quad (2.4.1)$$

$$a_j = \sum_{i=0}^{n-1} \left( \prod_{\alpha=0}^{r-1} \gamma_{m_\alpha}^{+i_\alpha a_\alpha} \right) A_i ; \quad j \in Z_n \quad (2.4.2)$$

Equations (2.4.1) and (2.4.2) are known as forward and inverse transforms respectively. They form Generalized Walsh-Hadamard transform pair.

If the sequence  $a$  is from  $\mathbb{R}^n$ , then the samples of the transform sequence  $A$  satisfy the following relationship, generally known as *complex-conjugacy relation* between the transform coefficients.

$$A_i = A_{n\ominus i}^* ; \quad * \text{ denotes complex-conjugate} \quad (2.4.3)$$

where  $n\ominus i = \langle (m_0 - i_0)_{m_0} (m_1 - i_1)_{m_1} \dots (m_{r-1} - i_{r-1})_{m_{r-1}} \rangle$

The set of transform coefficients  $\{A_i, A_{n\ominus i}\}$  is called a conjugacy class. A conjugacy class consists of a single real-valued element whenever  $i = n\ominus i$ .

Let  $\mathbb{C}_r^n$  denote the complex space restricted to the real images. It can be shown that GWHT is an isomorphism between  $\mathbb{R}^n$  and  $\mathbb{C}_r^n$ .

**Theorem 2.4.1:** Generalized Walsh-Hadamard transform is an isomorphism between the space of  $n$ -tuples over  $\mathbb{R}$  and the subspace,  $\mathbb{C}_r^n$ , of complex space,  $\mathbb{C}^n$ , restricted to the real

images.

**Proof:** Proof follows from the properties of GWHT.  $\square$

Since GWHT is an isomorphism between  $\mathbb{R}^n$  and  $\mathbb{C}_r^n$ , the algebras of group  $G$  over  $\mathbb{R}$  and  $\mathbb{C}_r^n$  are also isomorphic. Let us denote the group algebra of  $G$  over complex field,  $\mathbb{C}_r^n$  by  $CG_r$ . Transform vector  $A \in \mathbb{C}_r^n$  is the corresponding element  $A(X)$  of  $CG_r$ , where

$A(X) = \sum_{i=0}^{n-1} A_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right)$  where  $A_i \in \mathbb{C}_r^n$  and  $X_{\alpha}^i \in C_{m_{\alpha}}$ , a cyclic group of order  $m_{\alpha}$  with generator  $X_{\alpha}$ .

We now proceed to discuss the concept of ideal in  $CG_r$  [6].

**Theorem 2.4.2:** In the transform domain, the set of elements (polynomials) of  $CG_r$  having a common set of coefficients which are zero constitute an ideal in  $CG_r$ .

**Proof:** Let  $I$  be the set of polynomials of  $CG_r$  which have  $i_1, i_2, \dots, i_s^{\text{th}}$  coefficients as zero, where  $i_1, i_2, \dots, i_s \in \mathbb{Z}_n$ . First we show that  $I$  is a subgroup of  $CG_r$  under addition.

Let  $A(X) = \sum_{j=0}^{n-1} A_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right)$  and  $B(X) = \sum_{j=0}^{n-1} B_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right)$  belong to  $I$ . i.e.,  $A_j = B_j = 0$  for  $j \in \{i_1, i_2, \dots, i_s\}$ . Also let  $C(X) = A(X) + B(X)$  where  $C(X) = \sum_{j=0}^{n-1} C_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right)$ .

For any  $j \in \{i_1, i_2, \dots, i_s\}$ ,  $C_j = A_j + B_j = 0 + 0 = 0$ . Hence  $C(X) \in I$ . i.e.,  $I$  is closed under addition. Polynomial with all its coefficients as zero is in  $I$  which is the identity. The inverse of any element  $A(X)$  in  $I$  is  $(-A(X))$  which is also in  $I$  and associativity axiom is also satisfied. Hence  $I$  is subgroup of  $CG_r$  under addition.

Now, let  $D(X) = \sum_{j=0}^{n-1} D_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right)$  be an arbitrary element in  $CG_r$ . Also let  $C(X) = \sum_{j=0}^{n-1} C_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right) = \sum_{j=0}^{n-1} A_j D_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right)$ . Since  $A(X) \in I$ ,  $A_j = 0$  for all  $j \in \{i_1, i_2, \dots, i_s\}$ , which means  $A_j D_j = 0$  for all  $j \in \{i_1, i_2, \dots, i_s\}$ . i.e.  $C_j = 0$  for all  $j \in \{i_1, i_2, \dots, i_s\}$ .

i.e.  $C(X) \in I$ . Hence  $I$  is an ideal in  $CG_r$ .  $\square$

**We now give the equivalent description of minimal ideal, idempotent and minimal idempotent in the transform-domain**

**Minimal ideal:** The set of elements of  $CG_r$  having a common set of coefficients which are non-zero over a single conjugacy class forms a minimal ideal in  $CG_r$ .

**Idempotent:** An element  $E(X) = \sum_{j=0}^{n-1} E_j \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^j \right) \in CG_r$  is called an idempotent if it satisfies the following relationship.

$$E_j E_j = E_j \quad (2.4.4)$$

**Minimal idempotent:** An element of  $CG_r$  is having its coefficients as 1 s over a single conjugacy class and all other coefficients as 0 s is a minimal idempotent generator.

Let  $t$  be the number of conjugacy classes in  $CG_r$ . Denote the element of  $CG_r$  which is having its coefficients as 1 s over an  $i^{\text{th}}$  conjugacy class and other coefficients as 0 s by  $E^i(X)$ . This  $E^i(X)$  is nothing but the  $i^{\text{th}}$  minimal idempotent generator of the  $i^{\text{th}}$  minimal ideal,  $\langle E^i(X) \rangle$ . There will be as many minimal ideals as the number of conjugacy classes. Therefore  $CG_r$  can be expressed in the following form as the direct sum of minimal ideals.

$$CG_r = \langle E^0(X) \rangle \oplus \langle E^1(X) \rangle \oplus \dots \oplus \langle E^{t-1}(X) \rangle \quad (2.4.5)$$

Any other ideal in  $CG_r$  can be expressed as a direct sum of subset of these minimal ideals.

The idempotent generators  $E^i(X)$  of  $\langle E^i(X) \rangle$  satisfy the following properties:

$$\sum_{i=0}^{t-1} E_j^i = 1 ; j \in Z_n \quad (2.4.6)$$

$$E_k^i E_k^j = 0 ; i \neq j, k \in Z_n \quad (2.4.7)$$

$$\langle E^i(X) \rangle \cap \langle E^j(X) \rangle = 0 ; i \neq j \quad (2.4.8)$$

Before giving examples to illustrate above concepts, we digress to a discussion of the expression for the number of conjugacy classes in different real group algebras.

**Number of conjugacy classes in  $CG_r$ :**

For the purpose of calculating the number of conjugacy classes, we shall introduce the following artifice which will be helpful. It should be noted that the length of the



conjugacy class is either 1 or 2. If a group algebra has  $s$  single element conjugacy classes and  $d$  two element conjugacy classes, we shall represent the expression for the number of conjugacy classes as  $(sS + dD)$ . The total number can be obtained by putting  $S = 1$  and  $D = 1$ . If there are two or more cyclic groups, then the number of conjugacy classes is obtained as follows:

1) Symbolically multiply the expressions corresponding cyclic groups as polynomials in  $S$  and  $D$ .

2) Evaluate the above polynomial in  $S$  and  $D$  using the following rules.

$$S.S = S.D = D.S = 1 ; D^r = 2^{r-1} ; S^i D^j = D^j = 2^{j-1}.$$

The reason for the above rule is that the product of two cyclic conjugacy classes both of length 2 produces two conjugate classes of abelian group algebra each of length 2. In the rest of the cases only one conjugate class will be produced and the length of the conjugacy class is the product of the lengths of individual cyclic conjugate classes. In the following we give closed form expressions for some cases.

1) When  $G$  is a cyclic group of order  $n$

Number of conjugacy classes  $(t) = (2S + ((n/2)-1)D)$  for  $n$  even

(2.4.9)

$$= (S + ((n-1)/2)D) \text{ for } n \text{ odd.}$$

(2.4.10)

As explained earlier  $S$  and  $D$  refers to the conjugacy classes of lengths one and two respectively. For example, for  $n = 9$ , there are 5 conjugacy classes out of which one is of length one and the rest four are of length 2.

2) When  $G$  is a direct product group,  $C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ , of order  $n$

$$\text{Number of conjugacy classes } (t) = \prod_{i=0}^{r-1} t_i \quad (2.4.11)$$

where  $t_i$  is the number of conjugacy classes of a component cyclic group  $C_{m_i}$ . In this product substitute 1 for  $D.S = S.D = S.S$  and 2 for  $D.D$  (In general  $D^r = 2^{r-1}$ ). That gives

the number of conjugacy classes in  $CG_r$ . To illustrate this consider the following example.

**Example 2.4.1:** For  $G = C_3 \otimes C_3$ ,  $t = ?$

$$t = t_0 t_1 = (S + D).(S + D) = S.S + 2 S.D + D.D = 1 + 2 + 2 = 5.$$

The following interesting theorem may have an important consequence in regard to the study of filters.

**Theorem 2.4.3:** For odd  $m$ , the number of conjugacy classes in group algebra  $CC_r$  of a cyclic group of order  $m^p$ , is same as the number of conjugacy classes in group algebra  $CG_r$  of group  $G$  where  $G$  is a direct product group of  $p$  cyclic subgroups, each of order  $m$ .

**Proof:** Let  $t_c$  and  $t_a$  denote the number of conjugacy classes in  $CC_r$  and  $CG_r$  respectively.

Using (2.4.10),  $t_c = (m^p + 1)/2$ . Let  $t$  be the number of conjugacy classes of  $C_m$  algebra.

Using (2.4.11) and binomial expansion,

$$t_a = [S + ((m-1)/2)D]^p = \sum_{j=0}^p {}^p C_j S^{p-j} (m'D)^j ; m' = (m-1)/2$$

Separating the term corresponding to  $j = 0$  and invoking the rules of evaluation, the above expression simplifies to

$$t_a = 1 + \sum_{j=1}^p {}^p C_j S^{p-j} (m'D)^j = 1 + \sum_{j=1}^p {}^p C_j (m'D)^j = 1 + (1/2) \sum_{j=1}^p {}^p C_j (m-1)^j$$

Since using binomial theorem  $\sum_{j=0}^p {}^p C_j (m-1)^j = m^p$ ,  $t_a$  is given by

$$t_a = 1 + (m^p - 1)/2 = (m^p + 1)/2 = t_c \text{ as required. } \square$$

In what follows, we consider some examples and give the characterization of ideals in the form of tables. It is to be noted that ideals in cyclic group algebra  $CC_r$  are expressed as polynomials in  $Z$ .

**Example 2.4.2:**  $n = 8 ; r = 1 ; t = 5 ; Z^8 = 1$

**Table 2.4.1:** List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in  $C_8$  Group Algebra.

Conjugacy class no. i	Minimal idempotent $E^i(Z)$	Form of elements of ideal generated by $E^i(Z)$
0	$E^0(Z)=1$	$a'_0$
1	$E^1(Z)=Z + Z^7$	$A_1 Z + A_1^* Z^7$
2	$E^2(Z)=Z^2 + Z^6$	$A_2 Z^2 + A_2^* Z^6$
3	$E^3(Z)=Z^3 + Z^5$	$A_3 Z^3 + A_3^* Z^5$
4	$E^4(Z)=Z^4$	$a'_4 Z^4$

**Example 2.4.3:**  $n=8 ; r=2 ; m_0=4, m_1=2 ; t=6 ; X_0^4 = X_1^2 = 1$

**Table 2.4.2:** List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in  $C_4 \otimes C_2$  Group algebra.

Conjugacy class no. / i	Minimal idempotent $E^i(X)$	Form of elements of ideal generated by $E^i(X)$
0	$E^0(X)=1$	$a'_0$
1	$E^1(X)=X_0 + X_0^3$	$A_{10} X_0 + A_{10}^* X_0^3$
2	$E^2(X)=X_0^2$	$a'_2 X_0^2$
3	$E^3(X)=X_1$	$a'_3 X_1$
4	$E^4(X)=X_1 X_0 + X_1 X_0^3$	$A_{41} X_1 X_0 + A_{41}^* X_1 X_0^3$
5	$E^5(X)=X_1 X_0^2$	$a'_5 X_1 X_0^2$

Example 2.4.4:  $n=32$  ;  $r=1$  ;  $t=17$  ;  $Z^{32} = 1$

Table 2.4.3: List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in  $C_{32}$  Group Algebra.

Conjugacy class no. $i$	Minimal idempotent $E^i(Z)$	Form of elements of ideal generated by $E^i(Z)$
0	$E^0(Z)=1$	$a'_0$
1	$E^1(Z)=Z + Z^{31}$	$A_1 Z + A_1^* Z^{31}$
2	$E^2(Z)=Z^2 + Z^{30}$	$A_2 Z^2 + A_2^* Z^{30}$
3	$E^3(Z)=Z^3 + Z^{29}$	$A_3 Z^3 + A_3^* Z^{29}$
4	$E^4(Z)=Z^4 + Z^{28}$	$A_4 Z^4 + A_4^* Z^{28}$
5	$E^5(Z)=Z^5 + Z^{27}$	$A_5 Z^5 + A_5^* Z^{27}$
6	$E^6(Z)=Z^6 + Z^{26}$	$A_6 Z^6 + A_6^* Z^{26}$
7	$E^7(Z)=Z^7 + Z^{25}$	$A_7 Z^7 + A_7^* Z^{25}$
8	$E^8(Z)=Z^8 + Z^{24}$	$A_8 Z^8 + A_8^* Z^{24}$
9	$E^9(Z)=Z^9 + Z^{23}$	$A_9 Z^9 + A_9^* Z^{23}$
10	$E^{10}(Z)=Z^{10} + Z^{22}$	$A_{10} Z^{10} + A_{10}^* Z^{22}$
11	$E^{11}(Z)=Z^{11} + Z^{21}$	$A_{11} Z^{11} + A_{11}^* Z^{21}$
12	$E^{12}(Z)=Z^{12} + Z^{20}$	$A_{12} Z^{12} + A_{12}^* Z^{20}$
13	$E^{13}(Z)=Z^{13} + Z^{19}$	$A_{13} Z^{13} + A_{13}^* Z^{19}$
14	$E^{14}(Z)=Z^{14} + Z^{18}$	$A_{14} Z^{14} + A_{14}^* Z^{18}$
15	$E^{15}(Z)=Z^{15} + Z^{17}$	$A_{15} Z^{15} + A_{15}^* Z^{17}$
16	$E^{16}(Z)=Z^{16}$	$a'_{16} Z^{16}$

ample 2.4.5:  $n=32$  ;  $r=2$  ;  $m_0=4$ ,  $m_1=8$  ;  $t=18$  ;  $X_0^4 = X_1^8 = 1$

Table 2.4.1: List of Non-zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in  $C_4 \otimes C_8$  Group Algebra.

Conjugacy class no. i	Minimal idempotent $E^i(X)$	Form of elements of ideal $\langle E^i(X) \rangle$
0	$E^0(X)=1$	$a'_0$
1	$E^1(X)=X_0 + X_0^3$	$A_1 X_0 + A_1^* X_0^3$
2	$E^2(X)=X_0^2$	$a'_2 X_0^2$
3	$E^3(X)=X_1 + X_1^7$	$A_3 X_1 + A_3^* X_1^7$
4	$E^4(X)=X_1 X_0 + X_1^7 X_0^3$	$A_4 X_1 X_0 + A_4^* X_1^7 X_0^3$
5	$E^5(X)=X_1 X_0^2 + X_1^7 X_0^2$	$A_5 X_1 X_0^2 + A_5^* X_1^7 X_0^2$
6	$E^6(X)=X_1^2 + X_1^6$	$A_6 X_1^2 + A_6^* X_1^6$
7	$E^7(X)=X_1^2 X_0 + X_1^6 X_0^3$	$A_7 X_1^2 X_0 + A_7^* X_1^6 X_0^3$
8	$E^8(X)=X_1^2 X_0^2 + X_1^6 X_0^2$	$A_8 X_1^2 X_0^2 + A_8^* X_1^6 X_0^2$
9	$E^9(X)=X_1^3 + X_1^5$	$A_9 X_1^3 + A_9^* X_1^5$
10	$E^{10}(X)=X_1^3 X_0 + X_1^5 X_0^3$	$A_{10} X_1^3 X_0 + A_{10}^* X_1^5 X_0^3$
11	$E^{11}(X)=X_1^3 X_0^2 + X_1^5 X_0^2$	$A_{11} X_1^3 X_0^2 + A_{11}^* X_1^5 X_0^2$
12	$E^{12}(X)=X_1^4$	$a'_{12} X_1^4$
13	$E^{13}(X)=X_1^4 X_0 + X_1^4 X_0^3$	$A_{13} X_1^4 X_0 + A_{13}^* X_1^4 X_0^3$
14	$E^{14}(X)=X_1^4 X_0^2$	$a'_{14} X_1^4 X_0^2$
15	$E^{15}(X)=X_1^5 X_0 + X_1^3 X_0^3$	$A_{15} X_1^5 X_0 + A_{15}^* X_1^3 X_0^3$
16	$E^{16}(X)=X_1^6 X_0 + X_1^2 X_0^3$	$A_{16} X_1^6 X_0 + A_{16}^* X_1^2 X_0^3$
17	$E^{17}(X)=X_1^7 X_0 + X_1 X_0^3$	$A_{17} X_1^7 X_0 + A_{17}^* X_1 X_0^3$

## 2.5 Transform Domain Classification of P-I Filters

In this section, we present the transform-domain classification of abelian P-I filters and cyclic P-I filters based on ideals in their respective group algebras. First we start with abelian P-I filters.

In the transform-domain, abelian P-I filter modifies the spectrum of an input signal as given below

$$B_i = S_i \cdot A_i \quad (2.5.1)$$

where  $B \xrightarrow[\text{FGWHT}]{\text{IGWHT}} b$ ,  $S \xrightarrow[\text{FGWHT}]{\text{IGWHT}} s$  and  $A \xrightarrow[\text{FGWHT}]{\text{IGWHT}} a$ .

Any abelian P-I filter vector,  $S \in C_r^n$  can be interpreted as a linear combination of vectors which are non-zero over a single conjugacy class. Let  $S^i$  denote a vector which is non-zero over an  $i^{\text{th}}$  conjugacy class. As there are  $t$  conjugacy classes  $S$  can be represented as

$$S = S^0 + S^1 + \dots + S^{t-1} \quad (2.5.2)$$

These  $S^i$  are nothing but the elements of minimal ideals generated by  $E^i(X)$ . Therefore, transform of an abelian P-I filter vector can be interpreted as direct sum of component vectors of minimal ideals and the set of abelian P-I filters whose transform vectors have zeros at a common set of locations forms an ideal in abelian group algebra.

**Remark 2.5.1:** If the transform vector of an abelian P-I filter is a binary valued (0 or 1) vector then that abelian P-I filter can be interpreted as an idempotent.

**Example 2.5.1:**  $n = 8$  ;  $r = 2$  ;  $m_0 = 4$ ,  $m_1 = 2$ ,  $t = 6$

General form of abelian P-I filter vector is  $S = (a'_0 \ A_1 \ a'_2 \ A_1^* \ a'_3 \ A_4 \ a'_5 \ A_4^*)$

Let  $S = S^0 + S^1 + S^2 + S^3 + S^4 + S^5$  where

$$S^0 = (a'_0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) ; S^0(X) = a'_0$$

$$S^1 = (0 \ A_1 \ 0 \ A_1^* \ 0 \ 0 \ 0 \ 0) ; S^1(X) = A_1 X_0 + A_1^* X_0^3$$

$$S^2 = (0 \ 0 \ a'_2 \ 0 \ 0 \ 0 \ 0 \ 0) ; S^2(X) = a'_2 X_0^2$$

$$S^3 = (0 \ 0 \ 0 \ 0 \ a'_3 \ 0 \ 0 \ 0) ; S^3(X) = a'_3 X_1$$

$$S^4 = (0 \ 0 \ 0 \ 0 \ 0 \ A_4 \ 0 \ A_4^*) ; S^4(X) = A_4 X_1 X_0 + A_4^* X_1 X_0^3$$

$$S^5 = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ a'_5 \ 0) ; S^5(X) = a'_5 X_1 X_0^2$$

$S^i(X)$  is a general element of the minimal ideal generated by an  $i^{\text{th}}$  idempotent generator.

For this example, we now illustrate how an abelian P-I filter can be interpreted as an idempotent. The minimal idempotent generators are

$$E^0 = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$$

$$E^1 = (0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0)$$

$$E^2 = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0)$$

$$E^3 = (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)$$

$$E^4 = (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1)$$

$$E^5 = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0).$$

Any binary valued transform vector of an abelian P-I filter is a combination of  $E^i$ .

We now take up the classification of cyclic P-I filters based on ideals in cyclic group algebra. In the transform-domain, a cyclic P-I filter modifies the DFT spectrum of an input signal in accordance with the following equation

$$B_i = S_i \cdot A_i \quad (2.5.3)$$

$$\text{where } B \xrightarrow[\text{FDFT}]{\text{IDFT}} b, S \xrightarrow[\text{FDFT}]{\text{IDFT}} s \text{ and } A \xrightarrow[\text{FDFT}]{\text{IDFT}} a$$

The filter vector can be interpreted as direct sum of component vectors of the minimal ideals of cyclic group algebra  $CC_r$  and the set of cyclic filters whose DFT vectors have zero-valued components at a common set of locations is an ideal. Here also, a cyclic filter whose transform vector is a binary valued vector is an idempotent.

**Example 2.5.2:**  $n = 8$  ;  $r = 1$  ;  $t = 5$

General form of cyclic P-I filter vector is  $S = (a'_0 \ A_1 \ A_2 \ A_3 \ a'_4 \ A_3^* \ A_2^* \ A_1^*)$

Let  $S = S^0 + S^1 + S^2 + S^3 + S^4$  where

$$S^0 = (a'_0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) ; S^0(Z) = a'_0$$

$$S^1 = (0 \ A_1 \ 0 \ 0 \ 0 \ 0 \ 0 \ A_1^*) ; S^1(Z) = A_1 Z + A_1^* Z^7$$

$$S^2 = (0 \ 0 \ A_2 \ 0 \ 0 \ 0 \ A_2^* \ 0) ; S^2(Z) = A_2 Z^2 + A_2^* Z^6$$

$$S^3 = (0 \ 0 \ 0 \ A_3 \ 0 \ A_3^* \ 0 \ 0) ; S^3(Z) = A_3 Z^3 + A_3^* Z^5$$

$$S^4 = (0 \ 0 \ 0 \ 0 \ a'_4 \ 0 \ 0 \ 0) ; S^4(Z) = a'_4 Z^4$$

$S^i(Z)$  is a general element of the minimal ideal generated by  $i^{\text{th}}$  idempotent generator.

For this example, we now illustrate how a cyclic P-I filter can be interpreted as an idempotent. The minimal idempotent generators are

$$E^0 = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$$

$$E^1 = (0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1)$$

$$E^2 = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0)$$

$$E^3 = (0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0)$$

$$E^4 = (0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)$$

Any binary valued transform vector of a cyclic P-I filter is a combination of  $E^i$ .



# CHAPTER 3

## CHARACTERIZATION OF CYCLICALLY CLOSED ABELIAN P-I FILTERS

So far, we have discussed classification of P-I filters based on the ideals in group algebra. Consequently, cyclic P-I filters belong to the ideals in cyclic group algebra and abelian P-I filters belong to the ideals in abelian group algebra. Here, we present an important work of this thesis. We consider the characterization of P-I filters that have the structures of both cyclic as well as abelian filters inherently. Such P-I filters are closed under both cyclic and abelian permutations. We call this type of P-I filters as cyclically closed abelian P-I filters. MRX mapping between abelian and cyclic group algebras provides a basis for the characterization of these filters. Whenever the dimension of a P-I filter is a product of pairwise relatively prime integers, we consider the mapping based on CRT also and show that under this mapping any cyclic P-I filter can be realized equivalently as an abelian P-I filter. In what follows, we discuss the significance of this study in the implementation of cyclic P-I filters.

Concurrency or parallelism inherent to the structure of abelian class of P-I filters is more than that of cyclic class of P-I filters [1]. Therefore this can be exploited in the abelian P-I filter implementation of a cyclic P-I filter, in applications such as real-time signal processing where high computational throughput is required. And also, the transform-domain implementation of a cyclic P-I filter using GWHT requires lesser memory space than that is required for DFT. For example, to implement a cyclic P-I filter of length  $n$ , if we use DFT it requires  $n^2$  memory locations for storing transform matrix, whereas, GWHT, with factors  $m_i$  requires  $\sum m_i^2$  memory locations which is far lesser than that of DFT. Apart from these two advantages this study answered the questions like, by

using GWHT, is it possible to extract all the features those can be extracted by using DFT ? , if not all, then what are those cyclic features that GWHT can extract ? .

The problem is approached as follows. For a given class of abelian P-I filters, we characterize all the elements of abelian group algebra whose cyclic permutations are same as the corresponding abelian permutations. This correspondence is established on the basis of MRX mapping. Because of the necessity that these elements should belong to the ideals of abelian group algebra that are closed under cyclic permutations, firstly, we completely characterize all the ideals that are closed under cyclic permutations in the abelian group algebra. The converse of the above that whether any element of a cyclically closed abelian ideal is also a cyclically closed abelian P-I filter, may not be true always. This fact will be illustrated later with reasons. Secondly, since all the elements of cyclically closed ideals in abelian group algebra need not have the property that the effect of cyclic permutations are same as that of abelian permutations, we characterize all those elements which are cyclically closed abelian P-I filters.

The organization of this chapter is as follows. In Section 3.1, we describe cyclically closed abelian P-I filters. In Section 3.2, we interpret cyclically closed abelian P-I filters as elements of cyclically closed ideals in abelian group algebra. In Section 3.3, a theorem concerning the characterization of ideals in abelian group algebra that are closed under cyclic permutations is given. In Section 3.4, characterization of cyclically closed abelian P-I filters of a class of abelian P-I filters is presented in the form of a theorem. In Section 3.5, transform domain characterization of cyclically closed abelian filters of a class of abelian filters is given. In Section 3.6, the notion of a 2-D P-I filter is explained along with a procedure for identifying 2-D cyclically closed abelian filters. Finally in Section 3.7, we consider cyclic mappings based on CRT and show that under this mapping cyclic and abelian group algebras are isomorphic and that a cyclic P-I filter whose dimension is a product of pairwise relatively prime integers can be realized as an equivalent abelian P-I filter.

### 3.1 Cyclically Closed Abelian P-I Filters

In what follows, the concept of a cyclically closed abelian P-I filter is explained by means of an example.

Let us consider a P-I filter defined with respect to the cyclic permutation group  $PC_8$  of order 8 and another with respect to an abelian permutation group  $PG_8$ , which is defined with respect to the mixed-radix set  $M = \{ m_0 = 4, m_1 = 2 \}$ , of order 8. The former filter is represented by an  $8 \times 8$  matrix  $S_C$  and the latter by another  $8 \times 8$  matrix  $S_A$ .

Consider a vector  $s = (s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7) \in \mathbb{R}^8$  to represent zero<sup>th</sup> column of  $S_C$  and  $S_A$ .

Any  $k^{\text{th}}$  column of  $S_C$ , ( $k \in \mathbb{Z}_8$ ), is obtained by applying  $k^{\text{th}}$  permutation,  $p_k$ , of  $PC_8$  to  $s$  as given below.

$$p_k(s) = (s_{0\theta k} s_{1\theta k} \dots s_{i\theta k} \dots s_{7\theta k}) \quad (3.1.1)$$

where  $(i\theta k)$  is  $(i-k)$  modulo 8.

In a similar manner, the  $k^{\text{th}}$  column of  $S_A$  is obtained by applying  $k^{\text{th}}$  permutation,  $p_k$  of  $PG_8$  to  $s$  as shown below.

$$p_k(s) = (s_{0\theta k} s_{1\theta k} \dots s_{i\theta k} \dots s_{7\theta k}) \quad (3.1.2)$$

where  $i\theta k = \langle (i_0 - k_0)_4 (i_1 - k_1)_2 \rangle$ .

Consider a vector of form  $(a_1 \ 0 \ 0 \ 0 \ a_2 \ 0 \ 0 \ 0) \in \mathbb{R}^8$ . For this, we write the P-I filter matrices  $S_C$  and  $S_A$ , using the above (3.1.1) and (3.1.2).

$$S_A = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \end{bmatrix}$$

$$S_C = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \end{bmatrix}$$

It can be seen from the above that both  $S_C$  and  $S_A$  are identical. Therefore, other columns of any abelian P-I filter matrix with the zero<sup>th</sup> column of the above type can be obtained by applying permutations belonging to either  $CC_8$  or  $CG_8$ . That means this abelian P-I filter is also closed under the permutations of  $CC_8$ . Hence, this type of filters are called *Cyclically closed abelian P-I filters*. We now formally define cyclically closed abelian P-I system.

**Definition 3.1.1:** A system,  $T$ , which is defined invariant relative to a pair of cyclic transitive abelian permutation group,  $PC$ , of degree  $n$  and a transitive abelian permutation group,  $PG$ , of degree  $n$ , whose elements,  $p_k$ ,  $k \in Z_n$ , are being specified by the mixed-radix set  $\{m_0, m_1, \dots, m_{r-1}\}$ , is characterized by either of the following convolutional relationships.

$$b_i = \sum_{j=0}^{n-1} s_{i\theta j} a_j \quad (3.1.1)$$

where  $i\theta j = \langle (i_0 - j_0)_{m_0} (i_1 - j_1)_{m_1} \dots (i_{r-1} - j_{r-1})_{m_{r-1}} \rangle$

Alternatively, (3.1.1) can also be expressed in the form of matrix-equation as

$$b = Sa \quad (3.1.2)$$

where  $a$  and  $b$  are  $n \times 1$  matrices and  $S$  is  $n \times n$  matrix. Elements of  $S$  are given by

$$S_{i,j} = s_{i\theta j} \quad (3.1.3)$$

where  $i\theta j$  is as defined above.

$$b_i = \sum_{j=0}^{n-1} s_{(i-j) \bmod n} a_j \quad (3.1.4)$$

where  $a \in \mathbb{R}^n$  and  $s \in \mathbb{R}^n$

Alternatively, (3.1.4) can also be expressed in the form of matrix-equation as

$$b = Sa \quad (3.1.5)$$

where  $S$  is  $n \times n$  matrix whose elements are given by

$$S_{i,j} = s_{(i-j) \bmod n} \quad (3.1.6)$$

Let us denote the system which is defined invariant relative to both cyclic and abelian permutation groups of degree  $n$  by  $T_c$ . As the system,  $T_c$ , maps the space of real  $n$ -tuples,  $\mathbb{R}^n$ , into the space of real  $n$ -tuples, in which each of the  $n$ -tuples is a linear combination of the column vectors of  $S$ , the output signal space of the system  $T_c$  is closed under cyclic permutations of PC as well as abelian permutations of PG. Let us denote the subspace of  $\mathbb{R}^n$  closed under both cyclic as well as abelian permutations by  $\mathbb{R}_c^n$ . In this context, the system,  $T_c$ , which is a mapping from  $\mathbb{R}^n$  into subspace  $\mathbb{R}_c^n$  is called a cyclically closed abelian P-I filter. We now formally define cyclically closed abelian P-I filter.

**Definition 3.1.2:** A cyclically closed abelian P-I filter can be defined as a system  $T_c$ , which maps the space of real  $n$ -tuples,  $\mathbb{R}^n$ , into a linear subspace of  $\mathbb{R}^n$  which is closed under both cyclic permutations as well as abelian permutations belonging to PG and its input-output relationship is characterized either by (3.1.1) or by (3.1.4).

This cyclically closed abelian P-I filter is said to be *minimally cyclically closed abelian P-I filter* if its output signal space does not contain any proper subspaces closed under both cyclic permutations as well as abelian permutations. Figure 3.1.1 represents the sample domain interpretations of cyclically closed abelian P-I filter.

We now prove a lemma which will be used later.

**Lemma 3.1.1:** For a cyclically closed abelian P-I filter  $S_A - S_C = 0$ , where  $S_C$  and  $S_A$  are the cyclic and abelian P-I matrices associated with the filter.

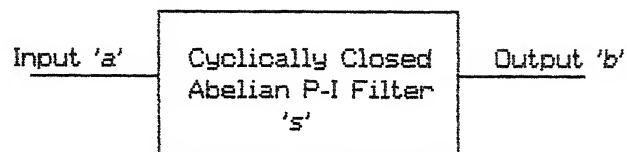


Figure 3.1.1.a

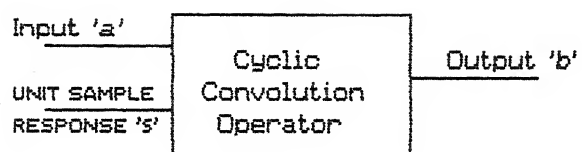


Figure 3.1.1.b

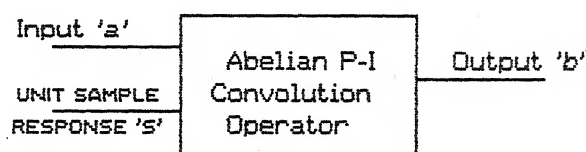


Figure 3.1.1.c

Figure 3.1.1

Sample Domain Interpretations of Cyclically Closed Abelian P-I Filter

**Proof:** Let  $s$  denote the unit-sample response of a cyclically closed abelian P-I filter. Any  $i, j$  th element of  $S_C$  is  $(S_C)_{i,j} = s_{(i-j) \bmod n}$  and that of  $S_A$  is  $(S_A)_{i,j} = s_{i \ominus j}$ . If an abelian P-I filter represented by  $S_A$  is invariant relative to PC then, any element of the matrix  $S_A$  should be  $(S_A)_{i,j} = s_{(i-j) \bmod n}$  which is  $(S_C)_{i,j}$ . Therefore for a cyclically closed abelian P-I filter  $s_{(i-j) \bmod n} = s_{i \ominus j}$  which means that  $S_A - S_C = 0$ .  $\square$

In the following section, we interpret cyclically closed abelian P-I filters as elements of cyclically closed ideals in abelian group algebra.

### 3.2 Cyclically Closed Abelian P-I Filters as Elements of Cyclically Closed Abelian Ideals

To explain about an ideal in abelian group algebra closed under cyclic permutations first we consider the interpretation of the operation of permutation on a vector as a multiplication of an element of a group algebra by a group element.

Let us consider a vector  $a = (a_0 \ a_1 \ \dots \ a_k \ \dots \ a_1 \ \dots \ a_{i+k} \ \dots \ a_{n-1}) \in \mathbb{R}^n$ . Consider

$a(X) = \sum_{i=0}^{n-1} a_i X^i$  as the element corresponding to  $a$  in group algebra  $RG$  of group  $G$ , whose

element  $X^i$  is indexed with respect to the mixed-radix set  $M = \{m_0 \ m_1 \ \dots \ m_{r-1}\}$  as

$X^i = x_{r-1}^{i_{r-1}} x_{r-2}^{i_{r-2}} \dots x_1^{i_1} x_0^{i_0} = \prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}}$  where  $i_{\alpha}$  is the index of  $i$  with respect to the

mixed-radix  $m_{\alpha}$  in the mixed-radix representation of  $i$  with respect to the mixed-radix set  $M = \{m_0 \ m_1 \ \dots \ m_{r-1}\}$ .

Let  $PG$  be the transitive abelian permutation group defined with respect to the mixed-radix set  $M$ . Denote an element belonging to  $PG$  by  $p_k$ .

Now we consider the effect of a permutation belonging to  $PG$  on  $a$ . An element  $p_k$ ,  $k \in \mathbb{Z}_n$ , permutes  $a$  into

$$a = (a_0 \ a_1 \ \dots \ a_k \ \dots \ a_1 \ \dots \ a_{i+k} \ \dots \ a_{n-1}) \quad (3.2.1)$$

$$p_k(a) = (a_{0 \oplus k} \ a_{1 \oplus k} \ \dots \ a_0 \ \dots \ a_{i \oplus k} \ \dots \ a_1 \ \dots \ a_{n-1 \oplus k}) \quad (3.2.2)$$

The corresponding element of  $RG$  is

$$P_k(a(X)) = \sum_{i=0}^{n-1} a_i X^{i+k} \quad (3.2.3)$$

Since  $X^{i+k} = X^i X^k$ , (3.2.3) can be written as

$$P_k(a(X)) = \sum_{i=0}^{n-1} a_i X^i X^k = X^k a(X) \quad ; \quad X^k \in G. \quad (3.2.4)$$

Therefore permuting an element  $a$  with  $P_k$  of  $PG$  is equivalent to multiplying the corresponding group algebra element with  $X^k$  of group  $G$ .

After having interpreted permutation as multiplication of a group element, now, we explain about cyclically closed ideal in group algebra as follows.

Cyclic permuted version of a vector can be obtained by treating it as a cyclic group algebra element and then multiplying it with a cyclic group element. Likewise its abelian permuted version can be obtained by treating it as an abelian group algebra element and then multiplying it with an abelian group element. Under mixed-radix (MRX) mapping  $X^k$  is mapped into  $Z^k$ , where  $X^k$  and  $Z^k$  are elements of abelian group and cyclic group respectively. If  $a(X)$  is an element of an ideal in abelian group algebra  $RG$ , then all  $X^i a(X)$  belong to this ideal. Now consider  $a(Z)$  obtained by applying MRX mapping to  $a(X)$ . Consider the set of all  $Z^k a(Z)$  ;  $k \in Z_n$ . This set forms an ideal in cyclic group algebra  $CG$ . Therefore this set is a cyclic ideal. Apply inverse MRX mapping to this set. If all the elements of this set belong to the ideal in abelian group algebra, then we call such an ideal as cyclically closed abelian ideal. This ideal is called minimally cyclically closed abelian ideal if it does not contain any sub ideals that are closed under both cyclic as well as abelian permutations. A space of  $n$ -tuples closed under both cyclic as well as abelian permutations forms a cyclically closed abelian ideal in a group algebra. For a vector  $a$  whose abelian permutation is same as cyclic permutation, multiplying abelian group algebra element,  $a(X)$ , with  $X^k$  is same as multiplying cyclic group algebra element,  $a(Z)$  with  $Z^k$ .

Since the output signal space of a cyclically closed abelian P-I filter has the



structure of cyclically closed abelian ideal, it can be interpreted in the following ways.

1) A cyclically closed abelian P-I filter is a mapping that maps any element of abelian group algebra  $RG$  into an element of a cyclically closed abelian ideal in  $RG$  and its input-output relationship is given by

$$b(X) = a(X) s(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), \dots, (X_{r-1}^{m_{r-1}} - 1) \quad (3.2.5)$$

where  $a(X)$ ,  $b(X)$  and  $s(X)$  are input, output and filter's unit-sample response elements respectively.

2) A cyclically closed abelian P-I filter is a mapping that maps any element of cyclic group algebra  $RG$  into an element of a cyclically closed abelian ideal in  $CG$  and its input-output relationship is given by

$$b(Z) = a(Z) s(Z) \bmod (Z^N - 1) \quad (3.2.6)$$

where  $a(Z)$ ,  $b(Z)$  and  $s(Z)$  are input, output and filter's unit-sample response elements respectively.

The fact that, the effects of cyclic permutations on unit-sample response vectors of cyclically closed abelian P-I filters are same as abelian permutations, these vectors should necessarily belong to cyclically closed abelian ideals. Therefore, cyclically closed abelian P-I filters can be interpreted as elements of cyclically closed abelian ideals. But, the converse that whether an element of a cyclically closed abelian ideal is also a cyclically closed abelian P-I filter, is not always true. This will be explained in Section 3.4. In the following section, we list out all cyclically closed ideals in abelian group algebra in the form of a theorem.

### 3.3 Identification of Cyclically Closed Ideals in Abelian Group Algebra

In Section 3.2 we have explained the meaning of a cyclically closed ideal in the

abelian group algebra. The relevant mapping is based on a mixed-radix system consisting of mixed radices corresponding to orders of component cyclic groups in the direct product of  $G$  (abbreviated MRX mapping). Under this mapping ' $g$ '  $\in G$ , an abelian group of order  $n$ , is mapped to a corresponding element of  $C_n$  that is associated with that particular element of  $Z_n$  that corresponds to the mixed-radix digit representation of ' $g$ ' and by linearity this notion is extended to MRX mapping of an element of the group algebra. Unlike the CRT mapping (which requires the component cyclic groups orders to be mutually prime ) this mapping is possible in any general abelian group algebra. Now we take up the characterization and identification of such cyclic abelian codes under MRX mapping [7].

First we consider an useful lemma based on the linearity of the MRX mapping that shows that given two ideals cyclic under MRX mapping then their sum is also cyclic under MRX mapping. This lemma simplifies the proof considerably in that it is enough to show some basic ideals are cyclic and the remaining cyclic ideals can be obtained as sums of these basic ideals. The lemma is proved for any general linear mapping ' $t$ ' and therefore in particular for MRX mapping.

**Lemma 3.3.1:** If  $C_1$  and  $C_2$  are cyclically closed ideals in an abelian group algebra under some linear mapping ' $t$ ', then ' $C_1 + C_2$ ' is also cyclically closed under ' $t$ '.

**Proof:** Any element of  $C_1 + C_2$  is of form  $a c_1 + b c_2$  where  $c_1 \in C_1$ ,  $c_2 \in C_2$ . Let  $z(a)$  denote cyclic shift of  $a$  by one place.  $(a)_i$  denote  $i^{\text{th}}$  bit of  $a$ .

Since ' $t$ ' is a linear mapping

$$\begin{aligned} t(a c_1 + b c_2) &= a t(c_1) + b t(c_2) \\ (z(t(a c_1 + b c_2)))_i &= (a t(c_1) + b t(c_2))_{(i-1) \bmod n} \\ &= (z(a t(c_1)))_i + (z(b t(c_2)))_i \end{aligned}$$

By hypothesis  $z(a t(c_1)) \in C_1$  and  $z(b t(c_2)) \in C_2$ . Therefore their sum belongs to

$$C_1 + C_2. \square$$

For the purpose of characterization of an abelian cyclic ideal, it is advantageous to introduce the notion of a cyclic ideal being minimally cyclic under MRX mapping. This should not be confused with the notion of a minimal ideal of an abelian group algebra. A minimally cyclic ideal may be a minimal or a non-minimal ideal of the abelian group algebra. This notion facilitates a different way of proving the characterization theorems given in [7]. Using this approach some fallacies in the proof given by [7] can also be taken care of. A formal definition can be given as follows:

**Definition 3.3.1:** An ideal  $M$  is said to be *minimally cyclic* under MRX mapping if it is cyclic and there exists no nontrivial sub ideal  $N$  of  $M$  that is cyclic under MRX mapping.

One immediate consequence is the following lemma which will be useful later.

**Lemma 3.3.2:** Suppose  $RG$  can be written as a direct sum of the following minimally cyclic ideals as shown below:

$$RG = \langle M_1 \rangle \oplus \langle M_2 \rangle \oplus \dots \oplus \langle M_t \rangle$$

Consider two non-minimal ideals, say  $M_i$  and  $M_j$ . Suppose  $N_i$  and  $N_j$  be any sub ideals of  $M_i$  and  $M_j$  respectively. Then  $N_i \oplus N_j$  is not cyclic under MRX mapping.

**Proof:** We prove the lemma by contradiction. Suppose  $N_i \oplus N_j$  is cyclic. Since  $M_i = N_i \oplus N_k$  is cyclic, it follows from Lemma 3.3.1 that the sum of this ideal with the ideal  $N_i \oplus N_j$ , i.e.,  $N_i \oplus N_k \oplus N_j = M_i \oplus N_j$ , is cyclic. Since  $M_i$  is cyclic, this implies that  $N_j$  is cyclic. This contradicts the minimality of  $M_j$ . Therefore  $N_i \oplus N_j$  can not be cyclic under MRX mapping.  $\square$

**Remark 3.3.1:** From above Lemmas 3.3.1 and 3.3.2 it is enough to find a direct sum decomposition of  $RG$  in terms of minimally cyclic ideals in order to characterize all the cyclic ideals in the abelian group algebra. For any nontrivial sub ideal or sum of nontrivial sub ideals of minimally cyclic ideals can not be cyclic according to Lemma 3.3.2. And once minimally cyclic ideals are known, any other cyclic abelian ideal can be obtained as sum of these minimally cyclic ideals according to Lemma 3.3.1.

We shall first consider when  $G$  is direct product of two cyclic groups. Once the basic

arguments are clearly understood it is easy to prove the results more generally for the case when  $G$  is direct product of  $r$  cyclic groups.

**Notations:**  $RG$  denotes an abelian group algebra of group  $G$  where  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes$

$C_{m_{r-1}}$ ;  $C_{m_i}$  is a cyclic group of order  $m_i$  with generator  $x_i$ . We shall represent  $\prod_{\alpha=0}^{r-1} x_{\alpha}^{i_{\alpha}}$

concisely by  $X^i$ , where ' $i$ ' is the mixed-radix representation of  $\langle i_0, i_1, \dots, i_{r-1} \rangle$  w.r.t

mixed-radix indices  $m_0, m_1, \dots, m_{r-1}$ . Let  $z^k[a(X)]$  represent the polynomial

corresponding to cyclic shifted vector of  $a(X)$  by  $k$  places. A minimal idempotent in the

$RC_{m_i}$  algebra is denoted by  $\theta(x_i)$  and the ideal generated by it is denoted by  $\langle \theta(x_i) \rangle$  and,

in particular  $j^{\text{th}}$  minimal ideal is denoted by  $\langle \theta_j(x_i) \rangle$ . Let  $R_0(x_i)$  represent the

idempotent consisting of all the elements of the group  $C_{m_i}$  each with the coefficient  $1/n$ .

For example,  $R_0(x)$  in  $RC_3$  algebra generated by  $x$ , is given by  $1/3(1 + x + x^2)$ . Number of

conjugate classes in the  $RC_{m_i}$  algebra is denoted by  $t_i$ .  $R_0(x_i)$  is taken as  $t_0^{\text{th}}$  ideal of  $RC_{m_i}$

algebra.

Let  $M_0 = 1$  and  $M_i = \prod_{k=0}^{i-1} m_k$ , for  $k = 1$  to  $r-1$ . Then MRX to cyclic mapping is

given by  $x_k \rightarrow z^{M_k}$ , for  $k = 0$  to  $r-1$ . In particular, for  $r = 2$  case, MRX to cyclic

mapping is given by  $x_0 \rightarrow z$  and  $x_1 \rightarrow z^{m_0}$ .

**Theorem 3.3.1:** An abelian group algebra  $RG$  with  $G = C_{m_0} \otimes C_{m_1}$  can be decomposed

into a direct sum of minimally cyclic ideals as follows:

$$RG = \langle \theta_0(x_1) \rangle \oplus \langle \theta_1(x_1) \rangle \oplus \dots \oplus \langle \theta_{t_1-1}(x_1) \rangle \oplus \\ \langle \theta_0(x_0)R_0(x_1) \rangle \oplus \dots \oplus \langle \theta_{t_0-1}(x_0)R_0(x_1) \rangle \oplus \langle R_0(x_0)R_0(x_1) \rangle$$

Any cyclic ideal under MRX mapping can be obtained as a sum of a subset of minimally cyclic ideals in the above direct sum.

**Proof:** We first show that every ideal in the direct sum decomposition given above is

cyclic. For this purpose we need the following equation from [7].

$$z^k[a(x_0, x_1)] - x_0^k a(x_0, x_1) = \sum_{j=0}^{m_1-1} \sum_{i=0}^{k-1} \{ a_{\langle (j-1)_{m_1}, (i-k)_{m_0} \rangle} - a_{\langle j, (i-k)_{m_0} \rangle} \} x_0^i x_1^j \quad (3.3.1)$$

For an ideal of the form  $\langle \theta_i(x_1) \rangle$ , its structure implies that  $\theta_{\langle j, 0 \rangle} = a \in R$ ; otherwise  $\theta_{\langle j, i \rangle} = 0$ . For  $k = 1$  to  $m_0 - 1$ , 'i' part of equation (3.3.1) is in the range 1 to  $m_0 - 1$  and therefore we have  $z^k[\theta_i(x_1)] = x_0^k \theta_i(x_1)$ . Since  $x_1^j \theta_i(x_1) = \theta_i^j(x_1)$  has a similar structure to  $\theta_i(x_1)$ , it follows that  $z^k[\theta_i^j(x_1)] = x_0^k \theta_i^j(x_1)$ . Consequently  $z^k[\theta_i(x_1)] = X^k \theta_i(x_1)$  for  $k = 0$  to  $m_0 m_1 - 1$ . Now any other element  $a \in \langle \theta_i(x_1) \rangle$  can be expressed as appropriate linear combinations of  $X^k \theta_i(x_1)$ , which are also  $z^k[\theta_i(x_1)]$ , for  $k = 0$  to  $m_0 m_1 - 1$  and therefore  $z(a) \in \langle \theta_i(x_1) \rangle$ .

For an ideal of the form  $\langle \theta_i(x_0) R_0(x_1) \rangle$ , the structure of its idempotent  $a$  implies that  $a_{\langle j, i \rangle} = a_{\langle j', i \rangle}$ ,  $j \neq j'$ . Therefore from equation (3.3.1) we have  $z^k[\theta_j(x_0) R_0(x_1)] = x_0^k \theta_j(x_0) R_0(x_1)$ , for  $k = 1$  to  $m_0 - 1$ . Since  $x_1^j \theta_j(x_0) R_0(x_1) = \theta_j(x_0) R_0(x_1)$  we have  $z^k[\theta_j(x_1)] = X^k \theta_j(x_1)$  for  $k = 0$  to  $m_0 m_1 - 1$ . Now arguing as above it can be shown that  $\langle \theta_j(x_0) R_0(x_1) \rangle$  is cyclic.

In order to prove the theorem as explained in Remark 3.3.1 We have to show that each  $M_i$  is minimally cyclic. There are both ideals of form  $\langle \theta_i(x_0) R_0(x_1) \rangle$  and  $\langle \theta_j(x_1) \rangle$ . Ideals of form  $\langle \theta_i(x_0) R_0(x_1) \rangle$  are easily seen to be minimal ideals of RG algebra. Therefore no nontrivial sub ideals can be considered in this case and consequently these ideals are minimally cyclic. Ideals of form  $\langle \theta_j(x_1) \rangle$ , however, may contain a nontrivial sub ideal. Let  $M$  be any nontrivial sub ideal of  $\langle \theta_j(x_1) \rangle$  that is cyclic under MRX mapping. We now deduce that such an ideal  $M$  is nothing but  $\langle \theta_j(x_1) \rangle$  itself. Suppose  $m(x_0, x_1)$  be the idempotent generator of  $M$ . Then from equation (3.3.1)

$$z(m(x_0, x_1)) - x_0 m(x_0, x_1) = m(x_1)$$

It will now be shown using the transform theory approach that  $m(x_1)$  is 0. Since the nonzeros of the ideal  $\langle \theta_j(x_1) \rangle$  are of the form  $(x_1^j + x_1^{m_1-j}) R_0(x_0)$ , the nonzeros of  $m(x_1) \in$

$\langle \theta(x_1) \rangle$  will also have this structure. On the other hand the nonzeros of  $M$ , a sub ideal of  $\langle \theta(x_1) \rangle$ , does not have a similar structure and therefore at least one conjugate class from  $(x_1^j + x_1^{m_1-j})R_0(x_0)$  is absent in the nonzeros of  $M$ . Let the zero positions of this conjugate class correspond to positions  $x_0^i x_1^j$  and  $x_0^{i'} x_1^{m_1-j}$ . As  $m(x_1) \in M$ , by hypothesis, its spectrum should have zeros in these positions. Since the nonzeros of  $m(x_1)$  are also of the form  $(x_1^j + x_1^{m_1-j})R_0(x_0)$ , the only way to satisfy these two conflicting conditions is to take  $m(x_1) = 0$ . This implies that  $m_{\langle i, n-1 \rangle} = m_{\langle j, n-1 \rangle}$  for  $i \neq j$ . Repeating the above arguments for the case  $x^j(m(x_0, x_1)) - x_0^j m(x_0, x_1)$ , for  $j = 2$  to  $m_1 - 1$ , it can be proved that  $m_{\langle i, j \rangle} = m_{\langle i', j \rangle}$  for  $j = 1$  to  $n-1$  and  $i \neq i'$ .

If not all of  $m_{\langle i, j \rangle}$ 's is zero, then  $m(x_0, x_1) = m'(x_0) R_0(x_1) + m'(x_1)$ . Since  $m'(x_0) R_0(x_1)$  belongs to some ideal of form  $\langle f(x_0) R_0(x_1) \rangle$ , which is orthogonal to  $M \subset \langle \theta_j(x_1) \rangle$ , it follows that  $m'(x_0)$  is zero. Since  $\langle \theta_j(x_1) \rangle$  corresponds to a minimal ideal of  $RC_{m_2}$  algebra,  $m'(x_1)$  should be equal to  $\theta_j(x_1)$ . Consequently,  $M$  is equal to  $\langle \theta_j(x_1) \rangle$  and therefore ideal  $\langle \theta_j(x_1) \rangle$  is minimally cyclically closed under MRX mapping.  $\square$

The following theorem considers the case when  $G$  is direct product of  $r$  cyclic groups.

**Theorem 3.3.2:** An abelian group algebra  $RG$  with  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$  can be decomposed into a direct sum of minimally cyclic ideals as follows:

$$RG = I_0 \oplus I_1 \oplus \dots \oplus I_{r-1}$$

where each group  $I_k$  can further be decomposed into

$$I_k = J_1^{(k)} \oplus J_2^{(k)} \oplus \dots \oplus J_{t_k-1}^{(k)} \quad (k = 1 \text{ to } r-1)$$

and

$$I_0 = J_1^{(0)} \oplus J_2^{(0)} \oplus \dots \oplus J_{t_0-1}^{(0)} \oplus \prod_{j=1}^{r-1} R_0(x_j)$$

where each  $J_i^{(k)}$  is an ideal of the form  $\langle \theta_i(x_k) \prod_{j=k+1}^{r-1} R_0(x_j) \rangle$  and  $t_k^{\text{th}}$  ideal is  $R_0(x_k)$  in the  $RC_{m_k}$  algebra. Any cyclic ideal under MRX mapping can be obtained as a sum of a subset

of minimally cyclic ideals in the above direct sum.

**Proof:** We prove the theorem by the method of induction on the number of cyclic groups 'r'. It is trivial for  $r = 1$  and true for  $r = 2$  from Theorem (3.3.1). Suppose it is true upto  $(r-1)$ . This implies that ideals that are minimally cyclic in group algebra  $RG'$  where  $G' = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-2}}$  are:

- 1) Ideals of the form  $\langle \theta(x_k) \prod_{j=k+1}^{r-1} R_0(x_j) \rangle$ , for  $k = 0$  to  $r-3$ .
- 2) Ideals of the form  $\langle \theta(x_{r-2}) \rangle$ .

In the case of  $G = G' \otimes C_{m_{r-1}}$  any ideal in the groups  $I_0$  to  $I_{r-2}$  are of the form  $\langle \theta(X')R_0(x_{r-1}) \rangle$ , where  $\langle \theta(X') \rangle$  is a minimally cyclic ideal from  $RG'$ . For any  $a(X) \in \langle \theta(X')R_0(x_{r-1}) \rangle$ , since  $x_{r-1}^j R_0(x_{r-1}) = R_0(x_{r-1})$ , it follows that  $a(X) = a(X')R_0(x_{r-1})$ , where  $a(X') \in \langle \theta(X') \rangle$ . In other words, the ideal  $\langle \theta(X')R_0(x_{r-1}) \rangle$  of  $RG$  is the ideal  $\langle \theta(X') \rangle$  of  $RG'$  repeated  $m_{r-1}$  times. Because of this  $(X'^j a(X')) = b(X')R_0(x_{r-1})$ , where  $X' \in G'$ , which can also be written as  $z^j[a(X')]R_0(x_{r-1})$  since  $\langle \theta(X') \rangle$  is cyclic. From this it follows that  $\langle \theta(X')R_0(x_{r-1}) \rangle$  is cyclic. Since  $\langle \theta(X') \rangle$  is minimally cyclic ideal by induction assumption, it follows that the ideals in the group  $I_0$  to  $I_{r-2}$  are minimally cyclic.

It remains to prove that the ideals of the form  $\langle \theta(x_{r-1}) \rangle$  are minimally cyclic. First we shall show that this type of ideal is cyclic. Structure of the idempotent  $\theta(x_{r-1})$  implies that  $\theta_{\langle i_{r-1}, 0, \dots, 0 \rangle} = a_k \in R$  ( $0 \leq i_{r-1} \leq m_{r-1}-1$ ); otherwise  $\theta_i = 0$ , where  $\langle i_{r-1}, i_{r-2}, \dots, i_0 \rangle$  is the mixed-radix representation of 'i' w.r.t. moduli  $m_0, m_1, \dots, m_{r-1}$ . Because of this structure of the idempotent  $\theta(x_{r-1})$  nonzero positions of  $(X^i \theta(x_{r-1}))$  and its corresponding cyclic shift  $z^i[\theta(x_{r-1})]$  tally and therefore  $(X^i \theta(x_{r-1})) = z^i[\theta(x_{r-1})]$  for  $i = 0$  to  $n-1$  ( $|G| = n$ ). From this it can be shown that  $\langle \theta(x_{r-1}) \rangle$  is cyclic as in Theorem 3.3.1.

We now show that the ideals of the form  $\langle \theta(x_{r-1}) \rangle$  are minimally cyclic under MRX mapping. If possible  $A \subset \langle \theta(x_{r-1}) \rangle$ , any nontrivial sub ideal, be cyclic under MRX

mapping. Let  $a(X)$  be the idempotent generator of ideal  $A$ . Our strategy is to show that  $x_0^i a(X)$  is same as  $z^i[a(X)]$  for  $i = 1$  to  $m_0 - 1$ . This implies that  $X^i a(X)$  is same as that of  $z^i[a(X)]$  for  $i = 0$  to  $n - 1$ , where  $n = \prod_{k=0}^{r-1} m_k$  is the order of  $G$ . From this the structure of  $a(X)$  can be known which will be later on used to show that  $\langle \theta_r(x_r) \rangle$  is minimally cyclic. Now to prove first  $z[a(X)] = x_0 a(X)$ , we introduce auxiliary polynomials  $a^0(X)$  to  $a^{r-2}(X)$  derived from  $a(X)$  as follows:

$$z[a(X)] - x_0 a(X) = a^0(x_1, x_2, \dots, x_{r-1}) = a^0(X) \quad (3.3.2)$$

$$z^{M_1}[a^0(X)] - x_2 a^0(X) = a^1(x_2, \dots, x_{r-1}) = a^1(X) \quad (3.3.3)$$

In general for  $i = 0$  to  $r-2$  we have

$$z^{M_i}[a^{i-1}(X)] - x_i a^{i-1}(X) = a^i(x_{i+1}, \dots, x_{r-1}) = a^i(X) \quad (3.3.4)$$

Finally for  $i = r-2$  we have

$$z^{M_{r-2}}[a^{r-3}(X)] - x_{r-2} a^{r-3}(X) = a^{r-2}(X) = a^{r-2}(x_{r-1}). \quad (3.3.5)$$

Note that in the above equations  $a^i(x_{i+1}, \dots, x_{r-1})$ , for  $i = 0$  to  $r-2$ , has nonzero locations only among the coordinates spanned by  $G^i = C_{m_{i+1}} \otimes C_{m_{i+2}} \otimes \dots \otimes C_{m_{r-1}}$ . Now we show that each of these polynomials  $a^0(X)$  to  $a^{r-2}(X)$  to be zero. Let us start from  $a^{r-2}(X)$  which is of the form  $a^{r-2}(x_{r-1})$ . By a spectral argument similar to the one used in Theorem 3.3.1,  $a^{r-2}(x_{r-1})$  is seen to be zero. Similarly when  $b^{r-3}(X) = x_{r-2} a^{r-3}(X)$  is used in place of  $a^{r-3}(X)$  in (3.3.4), we get, say  $b^{r-2}(x_{r-1})$ , which again by spectral argument can be shown to be zero. Continuing in this fashion  $m_{r-2} - 1$  times, it can be deduced that  $a^{r-3}(X)$  has the form  $a(x_{r-2})R_0(x_{r-1})$ . Since this belongs to an ideal of the form  $\langle f(x_{r-2})R_0(x_{r-1}) \rangle$  which is orthogonal to  $A \subset \langle \theta(x_{r-1}) \rangle$ , it follows that  $a^{r-3}(X)$  is equal to zero. This implies using (3.3.3), for  $i = r-3$

$$x_{r-3} a^{r-4}(X) = z^{M_{r-3}}[a^{r-4}(X)]$$

Now we show  $a^{r-4}(X)$  to be zero. As before using  $b^{r-4}(X) = x_{r-3} a^{r-4}(X)$  in place of  $a^{r-4}(X)$  in 3.3.3, for  $i = r-3$ , results in a polynomial  $a^r(X)$  that can be easily shown to



belong to the ideal of the form  $\langle g(x_{r-2})R_0(x_{r-1}) \rangle$ . Now invoking the orthogonality of this ideal with  $A \subset \langle \theta(x_{r-1}) \rangle$ ,  $a'(X)$  is seen to be zero. Therefore  $x_{r-3}^2 a^{r-4}(X) = z^{2.M} r-3 [a^{r-4}(X)]$ . In a similar way it can be proved that

$$x_{r-3}^k a^{r-4}(X) = z^{k.M} r-3 [a^{r-4}(X)], \text{ for } k = 1 \text{ to } m_{r-3} - 1$$

This implies  $a^{r-4}(X)$  to be cyclic among the coordinates  $G'' = C_{m_{r-3}} \times C_{m_{r-2}} \times C_{m_{r-1}}$ . Consequently  $a^{r-4}(X)$  is of the form  $a(x_{r-3})R_0(x_{r-2})R_0(x_{r-1})$  which belongs to an ideal orthogonal to ideal  $A$ . Therefore  $a^{r-4}(X)$  is zero. Continuing in this way it can be shown successively that  $a^{r-5}(X)$ , and so on up to  $a^0(X)$  are all zero polynomials, and finally we have  $z[a(X)] = (x_0 a(X))$ . Arguing on similar lines it can be shown that  $z^i[a(X)] = (x_0^i a(X))$  for  $i = 1$  to  $m_0 - 1$ .

Now all these results imply that the structure of  $a(X) = a(x_0) \prod_{k=1}^{r-1} R_0(x_k) + a(X')$ , where  $a(X')$  is among the coordinates spanned by  $C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ . By orthogonal property again  $a(x_0) = 0$ . Since  $a(X')$  is an idempotent in the group algebra consisting of direct product of  $r-1$  cyclic groups, invoking the induction assumption the structure of the idempotent  $a(X')$  is known. Again using the orthogonal property it can be seen that  $a(X') = f(x_{r-1})$ . Since  $\theta(x_{r-1})$  is minimal it follows that  $a(X) = f(x_{r-1}) = \theta(x_{r-1})$ . Consequently  $\langle \theta(x_{r-1}) \rangle$  is minimally cyclic under MRX mapping as was to be proved.  $\square$

The equivalent statement of Theorem 3.3.2 in the transform-domain is as follows.

In a group algebra  $CG_r$  of group  $G$ , where  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ , over the complex space restricted to the real images, ideals generated by the following idempotent generators are cyclic:

- 1) An all zero and an all 1 vectors.
- 2) Idempotent generators of form  $E^i(X_0)$ ;  $i = 0$  to  $t_0 - 1$

- 3) Idempotent generators of form  $E^i(X_j) \prod_{k=0}^{j-1} R_0(X_k)$  for  $i = 0$  to  $t_j-1$  and  $j = 1$  to  $r-1$ .
- 4) Linear combinations of above.

For the following examples, we characterize cyclically closed abelian ideals in the transform domain , using the above theorem.

**Example 3.3.1:**  $n = 8$  ;  $m_0 = 4$  ,  $m_1 = 2$

**Table 3.3.1:** List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in  $C_4 \otimes C_2$  Group Algebra.

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	$a'_0$
1	$X_0 + X_0^3$	$A_{10} X_0 + A_{11}^* X_0^3$
2	$X_0^2$	$a'_{20} X_0^2$
3	$X_1(1 + X_0 + X_0^3 + X_0^2)$	$a'_{31} X_1(a'_0 + A_{10} X_0 + A_{11}^* X_0^3 + a'_{20} X_0^2)$

**Example 3.3.2:**  $n = 32$  ;  $m_0 = 4$  ,  $m_1 = 8$

**Table 3.3.2:** List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in  $C_4 \otimes C_8$  Group Algebra.

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	$a'_0$
1	$X_0 + X_0^3$	$A_{10} X_0 + A_{11}^* X_0^3$
2	$X_0^2$	$a'_{20} X_0^2$
3	$(X_1 + X_1^7)(1 + X_0 + X_0^2 + X_0^3)$	$(A_{31} X_1 + A_{33}^* X_1^7)(a'_0 + A_{10} X_0 + a'_{20} X_0^2 + A_{11}^* X_0^3)$
4	$(X_1^2 + X_1^6)(1 + X_0 + X_0^2 + X_0^3)$	$(A_{41} X_1^2 + A_{44}^* X_1^6)(a'_0 + A_{10} X_0 + a'_{20} X_0^2 + A_{11}^* X_0^3)$
5	$(X_1^3 + X_1^5)(1 + X_0 + X_0^2 + X_0^3)$	$(A_{51} X_1^3 + A_{55}^* X_1^5)(a'_0 + A_{10} X_0 + a'_{20} X_0^2 + A_{11}^* X_0^3)$
6	$X_1^4(1 + X_0 + X_0^2 + X_0^3)$	$a'_{61} X_1^4(a'_0 + A_{10} X_0 + a'_{20} X_0^2 + A_{11}^* X_0^3)$

**Example 3.3.3:**  $n = 32$  ;  $m_0 = 4$  ,  $m_1 = 4$  ,  $m_2 = 2$

**Table 3.3.3:** List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in  $C_4 \otimes C_4 \otimes C_2$  Group Algebra

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	$a'_0$
1	$X_0 + X_0^3$	$A_{10} X_0 + A_{10}^* X_0^3$
2	$X_0^2$	$a'_2 X_0^2$
3	$(X_1 + X_1^3)(1 + X_0 + X_0^2 + X_0^3)$	$(A_{31} X_1 + A_{31}^* X_1^3)(a'_0 + A_{10} X_0 + a'_2 X_0^2 + A_{10}^* X_0^3)$
4	$X_1^2(1 + X_0 + X_0^2 + X_0^3)$	$a'_4 X_1^2(a'_0 + A_{10} X_0 + a'_2 X_0^2 + A_{10}^* X_0^3)$
5	$X_2(1 + X_0 + X_0^2 + X_0^3) \cdot (1 + X_1 + X_1^2 + X_1^3)$	$a'_5 X_2(a'_0 + A_{10} X_0 + a'_2 X_0^2 + A_{10}^* X_0^3) \cdot (a'_6 + A_{71} X_1 + a'_8 X_1^2 + A_{71}^* X_1^3)$

### 3.4 Characterization of Cyclically Closed Abelian P-I Filters

In this section we take up the characterization of cyclically closed abelian filters. As explained earlier, any cyclically closed abelian filter necessarily belongs to some ideal that is closed under MRX mapping. We have characterized all such cyclically closed ideals in a given abelian group algebra. However, not all of the elements belonging to a cyclically closed abelian ideal can be taken as cyclically closed abelian P-I filters. This is due to the fact (Lemma 3.1.1) that any such cyclically closed abelian P-I filter, say  $s$ , should have additional property that the cyclic P-I matrix  $S_C$  and abelian P-I matrix  $S_A$  associated with it must be the same. Let us consider the following example by way of illustration.

**Example 3.4.1:**  $n = 8$  ;  $m_0 = 4$  ,  $m_1 = 2$  ;  $x_0^4 = x_1^2 = 1$ .

Ideals closed under cyclic shifts are (1)  $\langle \theta(x_1) \rangle$  and (2)  $\langle \theta(x_0)R_0(x_1) \rangle$

Let us consider  $s = (a_0 \ 0 \ 0 \ 0 \ a_1 \ 0 \ 0 \ 0) \in \langle \theta(x_1) \rangle$ . In Section 3.1, we have seen that a P-I filter with this type of unit-sample response vector is a cyclically closed abelian P-I filter. Since  $s \in \langle \theta(x_1) \rangle$ , all the abelian permutations of  $s$  also belong to  $\langle \theta(x_1) \rangle$ . We now consider an abelian permutation of  $s$ ,  $x_0[s]$ , and show that it is not a cyclically closed abelian P-I filter.

$$x_0[s] = (0 \ a_1 \ 0 \ 0 \ 0 \ a_2 \ 0 \ 0)$$

Using (3.1.1) and (3.1.2), we write system matrices  $S_C$  and  $S_A$ .

$$S_A = \begin{bmatrix} 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

CENTRAL LIBRARY  
I. I. T., KANPUR

Acc. No. A110029

$$S_C = \begin{bmatrix} 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \\ a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

It can be seen that for  $x_0[s]$ ,  $S_C$  and  $S_A$  are not equal. Therefore not all the elements of cyclically closed abelian ideals are cyclically closed abelian P-I filters. This is because the concept of a cyclically closed ideal is less restrictive than that of a cyclically closed abelian P-I filter.

Alternatively, such cyclic filters from a cyclically closed abelian ideal form a subspace which exhibits algebra isomorphism with the corresponding cyclic group algebra. We now characterize such a subspace of a cyclically closed abelian ideal. First we consider the case when  $G = C_{m_0} \times C_{m_1}$  and later extend it to the case when  $G$  is direct product of  $r$  cyclic groups.

**Theorem 3.4.1:** In the group algebra  $RG$  with  $G = C_{m_0} \times C_{m_1}$  the following elements are cyclically closed abelian filters (c.c.a.filters):

- 1) All elements of the ideals of the form  $\langle \theta(x_0)R_0(x_1) \rangle$ .
- 2) All elements of the subspace consisting of elements of the form  $f(x_1)$  belonging to an ideal of the type  $\langle \theta(x_1) \rangle$ .
- 3) Elements obtained as linear combinations of the filters of above types.

part 3 is trivial since MRX to cyclic mapping is a linear mapping on group algebra.

$$\langle \theta(x_0)R_0(x_1) \rangle.$$

ing to the ideal is of the form  $a(x_0, x_1) = a(x_0, x_1)\theta(x_0)R_0(x_1)$ . Since

$x_1^j R_0(x_1)$  is same as that of  $R_0(x_1)$ , the above code word can be written as  $a'(x_0)R_0(x_1)$ , where  $a'(x_0)$  is an element in the ideal  $\langle \theta_0(x_0) \rangle$  and using equation (3.3.1), we have  $x_0^j a'(x_0)R_0(x_1) = z^j [a'(x_0)R_0(x_1)]$  for  $j = 1$  to  $m_0-1$ . Therefore it follows that  $X^k a'(x_0)R_0(x_1) = z^k [a'(x_0)R_0(x_1)]$  for  $k = 0$  to  $m_0 m_1 - 1$ . This shows that any element of the form  $a'(x_0)R_0(x_1) \in \langle \theta_0(x_0) R_0(x_1) \rangle$  qualify as c.c.a filters.

2) Ideals of the form  $\langle \theta(x_1) \rangle$ .

Note that the this ideal is generated by the following set  $S$  consisting of polynomials  $\{ (x_0^i x_1^j \theta(x_1) ; 0 \leq i \leq m_0-1, 0 \leq j \leq m_1-1) \}$ . Consider the polynomial  $(x_0^i x_1^j \theta(x_1))$  where  $i \neq 0$ . Then we have

$$x_0^{m_0-i} x_0^i x_1^j \theta(x_1) = x_1^j \theta(x_1)$$

But

$$z^{m_0-i} [x_0^i x_1^j \theta(x_1)] = z^{m_0} [x_1^j \theta(x_1)] = x_1^{j+1} \theta(x_1)$$

It is clear that the above two expressions may not be equal in general. This happens whenever  $i \neq 0$ . And when  $i = 0$ , that is for elements of form  $x_1^j \theta(x_1) = \theta'(x_1)$ , according to (3.3.1) satisfy  $X^k \theta'(x_1) = z^k [\theta'(x_1)]$ , for  $k = 0$  to  $m_0 m_1 - 1$ , and hence qualify as c.c.a filters. By linearity property of MRX mapping all elements of the subspace generated by the set  $S' = \{ \theta(x_1), x_1 \theta(x_1), \dots, x_1^{m_1-1} \theta(x_1) \}$  also have the property that the corresponding abelian and cyclic shifts are equal and hence qualify as c.c.a filters. Such an element is of the form  $f(x_1)$  as was to be proved.  $\square$

**Remark 3.4.1:** It should be noted that the subspaces spanned by ideals of type  $\langle \theta(x_0)R_0(x_1) \rangle$  and subspaces consisting of elements of the form  $f(x_1) \in \langle \theta(x_1) \rangle$  are isomorphic as a group algebra to the corresponding subspaces of the cyclic group algebra under MRX mapping. The structure of these subspaces indicate that the cyclic vector in the abelian case either a cyclic vector of  $C_{m_0}$  algebra repeated  $m_1$  times or a cyclic vector of  $C_{m_1}$  algebra spread out in the abelian group algebra such that the corresponding

polynomial is of the form  $f(x_1)$ .

Now we take up the characterization of c.c.a filters for the case when  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ .

**Theorem 3.4.2:** In the group algebra  $RG$  with  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$  the following elements are cyclically closed abelian filters:

- 1) All elements of the ideals of the type  $\langle \theta(x_0) \prod_{k=1}^{r-1} R_0(x_k) \rangle$ .
- 2) All elements of the subspace consisting elements of the form  $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)$  belonging to an ideal of the type  $\langle \theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \rangle$  for  $i = 1$  to  $r-1$ .
- 3) Elements obtained as linear combinations of the filters of above types.

**Proof:** We prove by induction on the number of cyclic groups in the direct product expansion of  $G$ . Theorem is trivial for  $r = 1$  and true for  $r = 2$  from Theorem 3.4.1. We suppose that the theorem is true for  $G$  consisting of  $r-1$  cyclic groups. If  $G' = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-2}}$  this implies that the following elements have c.c.a property in the  $RG'$  algebra:

Elements of the form  $f(x_i) \prod_{k=i+1}^{r-2} R_0(x_k)$  belonging to an ideal of the type  $\langle \theta(x_i) \prod_{k=i+1}^{r-2} R_0(x_k) \rangle$  for  $i = 0$  to  $r-2$ .

If  $G = G' \otimes C_{m_{r-1}}$  then as explained in Theorem 3.3.2 ideals are of the form

$\langle \theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \rangle$  and any element of the ideal is of the type  $(a(X') R_0(x_{r-1}))$ , where  $a(X')$  is a c.c.a filter by induction assumption. From this it is simple to verify that  $a(X') R_0(x_{r-1})$  is also a c.c.a filter. This accounts for all filter vectors mentioned in parts 1 and 2 except filters of the form  $f(x_{r-1})$ .

It remains to consider all possible filters that can be formed from the ideals of the



form  $\langle \theta(x_{r-1}) \rangle$ . It can be seen from the proof of Theorem 3.3.2 that all elements of the ideal of the type  $(x_{r-1}^j \theta(x_{r-1}))$ , for  $j = 0$  to  $m_{r-1}-1$ , have the corresponding cyclic and abelian shifts equal and therefore all elements of the subspace generated by these elements qualify as c.c.a filters.

Now consider an element of the form  $(X^{i^a} \theta(x_{r-1}))$  where  $X^{i^a} \in G'$ . Suppose 'i' be the first integer in the expansion of  $X^{i^a} = \prod_{i=0}^{r-2} x_i^{a_i}$  for which  $a_i \neq 0$  and let  $X^{i^a} = x_i^{a_i} X^{i^b}$ . Then  $x_i^{m_i - a_i} (X^{i^a} \theta(x_{r-1})) = X^{i^b} \theta(x_{r-1})$ . On the other hand,  $z^{M_i(m_i - a_i)} [X^{i^a} \theta(x_{r-1})] = z^{M_i m_i} [X^{i^b} \theta(x_{r-1})]$  which is not equal to  $X^{i^b} \theta(x_{r-1})$  in general. This can be seen as follows. If  $i = r-2$ , then  $z^{M_i m_i} [X^{i^b} \theta(x_{r-1})] = x_{r-2} \theta(x_{r-1})$  which may not be equal always. If  $i \neq r-2$ , then taking  $b = 0$ , it is easily seen that  $z^{M_i m_i} [\theta(x_{r-1})] = \theta(x_{r-1})$  is not always true. Therefore all possible c.c.a filters from the ideal of the form  $\langle \theta(x_{r-1}) \rangle$  are formed from the subspace consisting elements of the form  $f(x_{r-1})$ .  $\square$

**Remark 3.4.2:** As stated in Remark 3.4.1, in the general case also the subspaces spanned by the elements of the form  $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \in \langle \theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \rangle$  ( $i = 0$  to  $r-1$ ) are isomorphic as a group algebra to the corresponding subspaces of the cyclic group algebra under MRX mapping. The structure of these subspaces indicate that the cyclic vector in the abelian case either a cyclic vector of  $C_{m_0}$  algebra repeated  $\prod_{k=0}^{r-1} m_k$  times or a cyclic vector of  $C_{m_i}$  algebra spread out in the abelian group algebra such that the corresponding polynomial is of the form  $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)$  for  $i = 1$  to  $r-1$ .

### 3.5 Transform Domain Characterization of Cyclically Closed Abelian P-I Filters

Generalized Walsh Hadamard transform (GWHT) relates the Convolution operation

defined by the abelian permutation group, PG, in sample domain to point-wise multiplication in the transform domain. Whereas, Discrete fourier transform (DFT) relates the cyclic convolution operation in sample domain to point-wise multiplication in the transform domain. A system,  $T_c$ , that has the choice of being characterized by either of the convolutional relationships, can be described in the transform domain either by using DFT or by using GWHT. When the system,  $T_c$ , is used for manipulating the transform coefficients of input sequence, it can be viewed either as a cyclic filter (when DFT is used) or as an abelian filter (when GWHT is used). Figure 3.5.1 represents the transform domain interpretations of cyclically closed abelian P-I filter.

Theorem 3.4.2 gives all the elements in abelian group algebra whose cyclic permutations are same as abelian permutations. Its equivalent statement in the transform-domain is as follows.

In an abelian group algebra  $CG_r$  of group  $G$  where  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ , over the complex space restricted to the real images, the following are the cyclically closed abelian P-I filters:

- 1) All elements of the ideals of the type  $\langle E^i(X_0) \rangle$  for  $i = 0$  to  $t_0 - 1$ .
- 2) All elements of the subspace consisting of elements of the form  $\langle E^i(X_k) \rangle \prod_{j=0}^{k-1} R_0(X_j)$  belonging to an ideal of the type  $\langle E^i(X_k) \prod_{j=0}^{k-1} R_0(X_k) \rangle$  for  $i = 0$  to  $t_k - 1$  and  $k = 1$  to  $r - 1$ .
- 3) Elements obtained as linear combinations of filters of above types.

For the following examples, we list out all the cyclically closed abelian P-I filters in the transform domain.

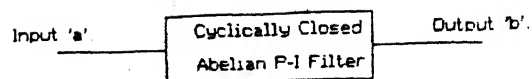


Fig. 3.5.1.a: Cyclically Closed Abelian P-I Filter

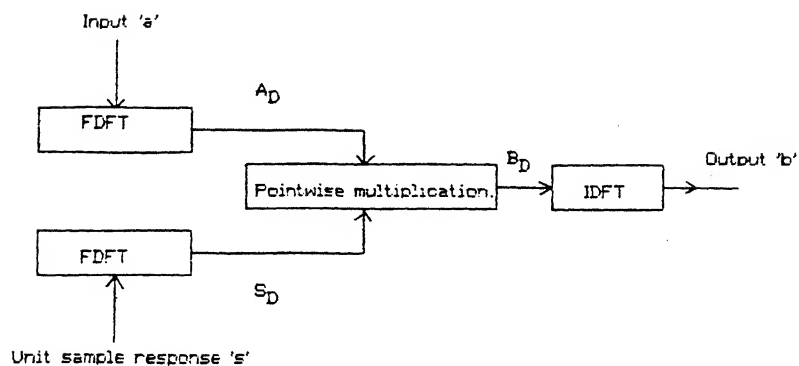


Fig. 3.5.1.b: Cyclically Closed Abelian P-I Filter as a Cyclic P-I Filter.

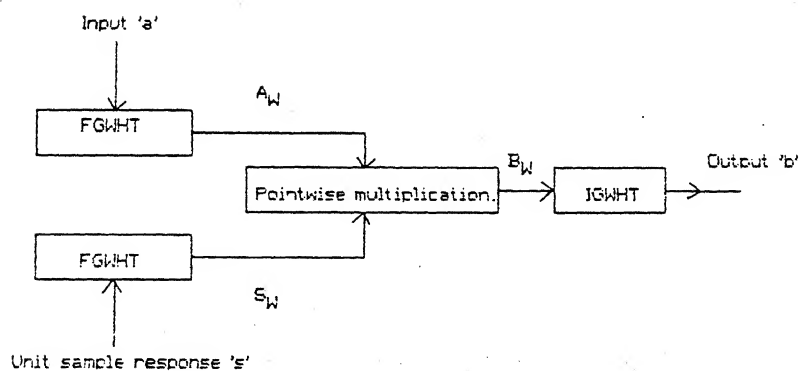


Fig. 3.5.1.c: Cyclically Closed Abelian P-I Filter as an Abelian P-I Filter

Fig. 3.5.1 - Transform Domain Interpretations of Cyclically Closed Abelian P-I Filter

**Example 3.5.1:**  $n = 8 ; r = 2 ; m_0 = 4 , m_1 = 2$

**Table 3.5.1:** List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P-I Filters in  $C_4 \otimes C_2$  Group Algebra.

Srl no:	Minimally cyclically closed ideal $\langle E^i(X) \rangle$	Form of elements of $\langle E^i(X) \rangle$ whose cyclic and abelian permutations are identical
0	$\langle 1 \rangle$	$a'_0$
1	$\langle X_0 + X_0^3 \rangle$	$A_{10} X_0 + A_{10}^* X_0^3$
2	$\langle X_0^2 \rangle$	$a'_2 X_0^2$
3	$\langle X_1(1 + X_0 + X_0^2 + X_0^3) \rangle$	$a'_3 X_1(1 + X_0 + X_0^2 + X_0^3)$

**Example 3.5.2:**  $n = 32 ; r = 2 ; m_0 = 4 , m_1 = 8$

**Table 3.5.2:** List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P-I Filters in  $C_4 \otimes C_8$  Group Algebra.

Srl no:	Minimally cyclically closed ideal $\langle E^i(X) \rangle$	Form of elements of $\langle E^i(X) \rangle$ whose cyclic and abelian permutations are identical
0	$\langle 1 \rangle$	$a'_0$
1	$\langle X_0 + X_0^3 \rangle$	$A_{10} X_0 + A_{10}^* X_0^3$
2	$\langle X_0^2 \rangle$	$a'_2 X_0^2$
3	$\langle (X_1 + X_1^7)(1 + X_0 + X_0^2 + X_0^3) \rangle$	$(A_{31} X_1 + A_{31}^* X_1^7)(1 + X_0 + X_0^2 + X_0^3)$
4	$\langle (X_1^2 + X_1^6)(1 + X_0 + X_0^2 + X_0^3) \rangle$	$(A_{41} X_1^2 + A_{41}^* X_1^6)(1 + X_0 + X_0^2 + X_0^3)$
5	$\langle (X_1^3 + X_1^5)(1 + X_0 + X_0^2 + X_0^3) \rangle$	$(A_{51} X_1^3 + A_{51}^* X_1^5)(1 + X_0 + X_0^2 + X_0^3)$
6	$\langle X_1^4(1 + X_0 + X_0^2 + X_0^3) \rangle$	$a'_6 X_1^4(1 + X_0 + X_0^2 + X_0^3)$

where  $p = \prod_{k=0}^{i-1} m_k$ ,  $A_{p+j}$  is a complex number and  $A_{p+j}^*$  is its complex conjugate.

Define  $R_0(X_k) = \sum_{i=0}^{m_i-1} X_k^i$ ;  $k = 0$  to  $r-1$ .

(4) Consider polynomials of following forms :

$$a) S^j(X_0) = A_j X_0^j + A_j^* X_0^{m_0-j}; j = 0 \text{ to } (t_0 - 1).$$

$$b) S^j(X_i) = \prod_{k=0}^{j-1} R_0(X_k); j = 0 \text{ to } (t_i - 1) \text{ and } i = 1 \text{ to } (r-1).$$

(5) Let  $S(X_0, X_1, \dots, X_{r-1})$  be the polynomial obtained by the linear combination of polynomials mentioned in Step 4. Also, let  $\langle i_0, i_1, \dots, i_{r-1} \rangle$  be the mixed-radix representation of 'i' w.r.t. M, in the sample domain. Now, map the polynomial  $S(X_0, X_1, \dots, X_{r-1})$  into a vector as  $S = (S_0 S_1 S_2 \dots S_i \dots S_{n-1})$  where  $S_i$  is the coefficient of the term  $\left( \prod_{\alpha=0}^{r-1} X_{\alpha}^{i_{\alpha}} \right)$  in  $S(X_0, X_1, \dots, X_{r-1})$ .

Each of these complex valued vectors,  $S = (S_0 S_1 S_2 \dots S_i \dots S_{n-1})$ , satisfying complex conjugacy relation can be taken as transform of the unit sample response vectors of cyclically closed abelian filters. Therefore, there are as many filters as the number of distinct vectors, S.

By way of illustration of the above procedure, we consider the following example.

**Example 3.5.4:**

$$(1) n = 8; m_0 = 4, m_1 = 2; X_0^4 = X_1^2 = 1$$

(2) conjugacy classes of  $m_0$  and  $m_1$  are  $(0)$ ,  $(1, 3)$  &  $(2)$  and  $(0)$  &  $(1)$  respectively. Number of conjugacy classes of  $m_0$  and  $m_1$  are  $t_0 = 3$  and  $t_1 = 2$  respectively.

(3) In polynomial form Conjugacy classes of  $m_0$  and  $m_1$  are

$$S^0(X_0) = a'_0, S^1(X_0) = A_1 X_0 + A_1^* X_0^3 \text{ \& } S^2(X_0) = a'_2 X_0^2 \text{ and } S^0(X_1) = a'_3 \text{ \& } S^1(X_1) = a'_4 X_1 \text{ respectively. (coefficients of single element conjugacy classes are reals)}$$

$$\text{Define } R_n(X_n) = 1 + X_n + X_n^2 + X_n^3.$$

(4) Consider polynomials of following forms:

(a)  $S^j(X_0)$  ;  $j = 0$  to  $3$ .

(b)  $S^j(X_1) \cdot R_0(X_0)$  ;  $j = 0$  to  $1$ .

(5)  $S(X_0, X_1)$  be the polynomial obtained by the linear combination of polynomials mentioned in Step 4 .

$$S(X_0, X_1) = a'_0 + (A_1 X_0 + A_1^* X_0^3) + (a'_2 X_0^2) + a'_4 X_1 (1 + X_0 + X_0^2 + X_0^3) = \\ a'_0 + A_1 X_0 + a'_2 X_0^2 + A_1^* X_0^3 + a'_4 X_1 + a'_4 X_1 X_0 + a'_4 X_1 X_0^2 + A_4 X_1 X_0^3$$

where  $A_i$  are complex coefficients.

This polynomial  $S(X_0, X_1)$  is mapped into a vector as  $S^n = (S_0 S_1 \dots S_i \dots S_7)$  where  $S_i$  is the coefficient of the term  $X_0^{i_0} X_1^{i_1}$  ;  $i_0$  and  $i_1$  mixed-radix representation of  $i$  in  $M$  in  $S(X_0, X_1)$ .

$$S = (a'_0 \ A_1 \ a'_2 \ A_1^* \ a'_4 \ a'_4 \ a'_4 \ a'_4)$$

**Remark 3.5.1:** In the sample domain by means of mixed-radix mapping we obtain the cyclic group algebra element for a given abelian group algebra element. But, in the transform domain this mapping is to be reversed for obtaining the corresponding cyclic group algebra element. This is illustrated in the example given below.

**Example 3.5.5:** Let  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$  ;  $n = \prod_{\alpha=0}^{r-1} m_\alpha$  ;  $X_\alpha^m = 1$  ;  $\alpha \in Z_r$  and let  $Z$  is a generator of a cyclic group,  $C_n$ , of order  $n$ . In the sample domain under MRX mapping  $X_\alpha^{i_\alpha}$  ;  $\alpha \in Z_r$  is mapped into  $Z^{i_\alpha M_\alpha}$  where  $M_\alpha = \prod_{k=0}^{\alpha-1} m_k$ . Whereas, in the transform domain  $X_\alpha^{i_\alpha}$  is mapped into  $Z^{i_\alpha M_\alpha}$  ;  $\alpha \in Z_r$  and  $M_\alpha = \prod_{k=\alpha+1}^{r-1} m_k$ .

In what follows, we give a procedure for finding the cyclic P-I filter for a given abelian P-I filter (Here, we are considering only those abelian filters that are closed under cyclic permutations).

Procedure for finding cyclic filter for a given abelian filter:

Let  $S(X) = \sum_{i=0}^{n-1} S_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right)$  be a cyclically closed abelian filter whose equivalent cyclic filter vector is to be found.

By applying the MRX mapping,  $X_{\alpha}^i = Z^{i M_{\alpha}}$ ;  $\alpha \in Z_r$  and  $M_{\alpha} = \prod_{k=\alpha+1}^{r-1} m_k$ , to  $S(X)$ , corresponding cyclic filter  $S(Z)$  can be obtained. Then take IDFT of  $S(Z)$ , or take IGWHT of  $S(X)$ . Denote it by  $s(X)$ . In the sample-domain, by substituting  $Z^{i M_{\alpha}}$  for  $X_{\alpha}^i$ ;  $\alpha \in Z_r$  and  $M_{\alpha} = \prod_{k=0}^{\alpha-1} m_k$  we can obtain  $s(Z)$ , where  $s(Z)$  is IDFT of  $S(Z)$ .

**Example 3.5.6:**  $n = 8$ ;  $m_0 = 4$ ,  $m_1 = 2$ ;  $X_0^4 = X_1^2 = Z^8 = 1$

Let  $S(X) = X_0 + X_0^3$  be the transform of a unit-sample response element of a cyclically closed abelian P-I filter whose equivalent cyclic P-I filter vector is to be found.

By applying transform domain MRX mapping,  $X_0 = Z^2$  and  $X_1 = Z$ , to  $S(X)$  we get the corresponding cyclic P-I filter vector  $S(Z)$  as

$$S(Z) = Z^2 + Z^6.$$

Its sample domain vector,  $s(Z)$ , is obtained by taking IDFT of  $S(Z)$ .

$$s(Z) = 1 - Z^2 + Z^4 - Z^6$$

The other way to find  $s(Z)$  is, first find the sample domain vector,  $s(X)$ , of  $S(X)$  (by taking IGWHT of  $S(X)$ ).

$$s(X) = 1 - X_0^2 + X_1 + X_1 X_0^2.$$

Applying sample domain MRX mapping,  $X_0 = Z$  and  $X_1 = Z^4$ , to  $s(X)$  we get the corresponding cyclic P-I filter vector  $s(Z) = 1 - Z^2 + Z^4 - Z^6$ .

### 3.6 A Procedure for Identifying 2-D Cyclically Closed Abelian P-I Filters

In this section, we proceed with the filtering of 2-D finite length discrete data on the lines of the facts discussed so far. To begin with, we review some of the concepts

related to 2-D P-I systems [9].

Here, the signal space of our interest is the space consisting of all  $n \times m$  matrices over reals. It can be treated as a vector space,  $V$ , of real  $n \times m$  matrices. A 2-D finite discrete system is a mapping of the type  $V \longrightarrow V$ . Let us take, for instance, an arbitrary input signal  $a$  from  $V$ , and permute the rows of ' $a$ ' by members of a transitive abelian permutation group  $PG_1$  of order  $m$  and the columns by another transitive abelian permutation group  $PG_2$  of order  $n$ . If the rows and columns of the corresponding output signal of the system  $T$  also gets permuted exactly in the same manner, then  $T$  is invariant to such permutations. Then we say that  $T$  is a 2-D permutation-invariant (2-D P-I) system relative to the pair of transitive abelian permutation groups  $PG_1$  and  $PG_2$ .

Thus, with a vector space of matrices over reals treated as a signal space, the operation of a generalized convolution admits of being treated as a 2-D P-I system relative to  $PG_1$  and  $PG_2$ . In other words a 2-D P-I system is described by:

$$b_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_{k\ominus i, l\bar{\vee} j} a_{i,j} ; k \in Z_n, l \in Z_m \quad (3.6.1)$$

where  $a$ ,  $b$  and  $s$  are respectively the input, output and systems unit sample response arrays;  $\bar{\vee}$  and  $\ominus$  respectively denote pointwise subtraction operation in two mixed-radix number systems, one with mixed-radix set  $M = \{m_0, m_1, \dots, m_{r-1}\}$  w.r.t which  $PG_1$  is defined and the other with mixed-radix set  $N = \{n_0, n_1, \dots, n_{s-1}\}$  w.r.t which  $PG_2$  is defined.

**Definition 3.6.1:** A 2-D P-I filter is a system  $T$  which maps the space of real  $n \times m$  matrices,  $V$ , into a linear subspace of  $V$  in which the rows and columns of the matrices are invariant relative to  $PG_1$  and  $PG_2$  respectively and its input-output relationship is characterized by (3.6.1).

Generally, we call this filter as a 2-D abelian P-I filter. If  $PG_1$  and  $PG_2$  are cyclic permutation groups, then, we call it as a 2-D cyclic P-I filter. If the effect of permutations



belonging to cyclic permutation groups  $PC_1$  (transitive cyclic permutation group of degree  $m$ ) and  $PC_2$  (transitive cyclic permutation group of degree  $n$ ) is same as that of permutations belonging to the permutation groups  $PG_1$  and  $PG_2$  on rows and columns of the unit-sample response matrix respectively, then the filter being characterized by this type of unit-sample response matrix is called a 2-D cyclically closed abelian P-I filter.

**Definition 3.6.2:** A 2-D cyclically closed abelian P-I filter is a system  $T$  which maps the space of real  $n \times m$  matrices,  $V$ , into a linear subspace of  $V$  in which the rows and columns of the matrices are invariant relative to the pair of permutation groups  $PC_1$  and  $PG_1$  and  $PC_2$  and  $PG_2$  respectively and its input-output relationship can be characterized either by (3.6.1) or by the following equation.

$$b_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_{(k-i) \bmod n, (l-j) \bmod m} a_{i,j} \quad ; k \in Z_n, l \in Z_m \quad (3.6.2)$$

This is a 2-D cyclic convolutional relationship which is a special case of the generalized convolution defined as above in (3.6.1).

Alternatively, 2-D P-I system can be described in the transform domain using 2-D GWHT as follows.

A 2-D GWHT defined with respect to the mixed-radix sets  $M = \{m_0, m_1, \dots, m_{r-1}\}$ ,  $N = \{n_0, n_1, \dots, n_{s-1}\}$  maps  $a \in V$  into a matrix of complex numbers,  $A$ , in the transform domain. These two matrices are related by the following equations.

$$A_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left( \prod_{a=0}^{r-1} \gamma_{n_a}^{-i a^k} \right) \left( \prod_{a=0}^{s-1} \gamma_{m_a}^{-j a^l} \right) a_{i,j} \quad ; k \in Z_n, l \in Z_m \quad (3.6.3)$$

where  $\gamma_{m_a}$  and  $\gamma_{n_a}$  are  $m_a^{\text{th}}$  and  $n_a^{\text{th}}$  roots of unity respectively.

$$a_{i,j} = (1/mn) \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left( \prod_{a=0}^{r-1} \gamma_n^i a^k \right) \left( \prod_{a=0}^{s-1} \gamma_m^j a^l \right) A_{k,l} ; i \in Z_n, j \in Z_m \quad (3.6.4)$$

Equations (3.6.3) and (3.6.4) are known as 2-D Generalized Walsh-Hadamard transform pair.

If the matrix  $a$  belongs to  $V$ , then the samples of the transform matrix  $A$  satisfy the following relationship, generally known as *complex-conjugacy relation* between the transform coefficients.

$$A_{i,j} = A_{n_{\Theta}i, m_{\nabla}j}^* \quad (3.6.5)$$

where  $*$  denotes complex-conjugate,  $\Theta$  and  $\nabla$  respectively denote pointwise subtraction operation in two mixed-radix number systems, one with mixed-radix set  $M = \{m_0, m_1, \dots, m_{r-1}\}$  and the other with mixed-radix set  $N = \{n_0, n_1, \dots, n_{s-1}\}$

2-D GWHT defined as in (3.6.3) and (3.6.4) replaces the convolution operation defined as in (3.6.1) with pointwise multiplication. Therefore, in the transform domain (3.6.1) can be expressed as

$$B_{i,j} = S_{i,j} \cdot A_{i,j} \quad (3.6.6)$$

where  $B \xrightarrow[\text{FGWHT}]{\text{IGWHT}} b$ ,  $S \xrightarrow[\text{FGWHT}]{\text{IGWHT}} s$  and  $A \xrightarrow[\text{FGWHT}]{\text{IGWHT}} a$

Now with this background and by making use of the procedure given in Section 3.5 for identification of cyclically closed abelian P-I filters, we will give a procedure for identification of 2-D cyclically closed abelian P-I filters.

**Procedure for identifying 2-D cyclically closed abelian P-I filters:**

- 1) For a given pair of ' $n, m$ ' choose mixed-radix sets  $N = \{n_0, n_1, \dots, n_i, \dots, n_{s-1}\}$  and  $M = \{m_0, m_1, \dots, m_i, \dots, m_{r-1}\}$ .
- 2) Find out the conjugacy classes for all  $m_i, 0 \leq i \leq r-1$ . Let  $t_i$  be the number of

conjugacy classes of an  $m_i$ .

3) Express each such conjugacy class as a polynomial in  $X_i$ . Denote the  $j^{\text{th}}$  conjugacy class of  $m_i$  by  $S^j(X_i)$ .

where  $S^j(X_i) = A_{p+j} X_i^j + A_{p+j}^* X_i^{m_i-j}$ ;  $j = 0$  to  $(t_i-1)$ ,  $i = 0$  to  $r-1$ ,  $p = \prod_{k=0}^{i-1} m_k$   
 $A_{p+j}$  is a complex number and  $A_{p+j}^*$  is its complex conjugate.

Define  $R_0(X_k) = \sum_{i=0}^{m_i-1} X_k^i$ ;  $k = 0$  to  $r-1$ .

4) Consider polynomials of following forms:

(a)  $S^j(X_0) = A_j X_0^j + A_j^* X_0^{m_0-j}$ ;  $j = 0$  to  $(t_0-1)$ .

(b)  $S^j(X_i) \cdot \prod_{k=0}^{j-1} R_0(X_k)$ ;  $j = 0$  to  $(t_i-1)$  and  $i = 1$  to  $(r-1)$ .

5) Let  $S(X_0, X_1, \dots, X_{r-1})$  be the polynomial obtained by the linear combination of polynomials mentioned in step 4. Also, let  $\langle i_0, i_1, \dots, i_{r-1} \rangle$  be the mixed-radix representation of 'i' w.r.t.  $M$ . Now, map the polynomial  $S(X_0, X_1, \dots, X_{r-1})$  into a vector as  $S = (S_0 S_1 S_2 \dots S_i \dots S_{n-1})$  where  $S_i$  is the coefficient of the term  $(\prod_{\alpha=0}^{r-1} X_{\alpha}^{i_{\alpha}})$  in  $S(X_0, X_1, \dots, X_{r-1})$ .

6) Repeat from Step 2 to Step 5 for the second mixed-radix set. Let us denote the vectors corresponding to the two mixed-radix sets  $M$  and  $N$  by  $S^m = (S_0^m S_1^m S_2^m \dots S_j^m \dots S_{m-1}^m)$  and  $S^n = (S_0^n S_1^n S_2^n \dots S_i^n \dots S_{n-1}^n)$  respectively.

7) Construct a matrix  $S$  using the following equation.

$$S_{i,j} = S_i^n \cdot S_j^m; i \in Z_n \text{ and } j \in Z_m \quad (3.6.7)$$

This matrix  $S$  is the transform of the unit sample response of a 2-D cyclically closed abelian P-I filter. For obtaining the corresponding cyclic P-I filter, first apply the mapping to the vectors  $S^n$  and  $S^m$  as mentioned in Example 3.5.5 and then construct the matrix as given in (3.6.7)

**Example 3.6.1:**

1)  $n = 8$ ,  $m = 8$ ,  $N = \{n_0 = 4, n_1 = 2\}$  and  $M = \{m_0 = 4, m_1 = 2\}$

First consider  $n$ ,  $N$ .

2) From Example 3.5.4, we obtain  $S^n = (a'_0 \ A_1 \ a'_2 \ A_1^* \ a'_4 \ a'_4 \ a'_4 \ a'_4)$  and

$$S^m = (b'_0 \ B_1 \ b'_2 \ B_1^* \ b'_4 \ b'_4 \ b'_4 \ b'_4)$$

3) Construct a matrix  $S$  using the following equation.

$$S_{i,j} = S_i^n \cdot S_j^m \quad ; i \in Z_8 \text{ and } j \in Z_8$$

This matrix  $S$  is the transform of the unit sample response of a 2-D cyclically closed abelian P-I filter.

The following is the general form of transform of the unit sample response matrix of a 2-D cyclically closed abelian P-I filter of dimension 8.

$$S = \begin{bmatrix} c'_0 & D_0 & a'_0 & D_0^* & b'_0 & b'_0 & b'_0 & b'_0 \\ C_1 & D_1 & A_1 & D_1^* & B_1 & B_1 & B_1 & B_1 \\ c'_2 & D_2 & a'_2 & D_2^* & b'_2 & b'_2 & b'_2 & b'_2 \\ C_1^* & D_1^* & A_1^* & D_1 & B_1^* & B_1^* & B_1^* & B_1^* \\ c'_3 & D_3 & a'_3 & D_3^* & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D_3^* & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D_3^* & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D_3^* & b'_3 & b'_3 & b'_3 & b'_3 \end{bmatrix}$$

### 3.7 Equivalent Cyclic P-I Filters based on CRT Mapping

In the above discussion, we have shown that under MRX mapping only a part of abelian group algebra, namely vector space formed by cyclically closed abelian P-I filters, is isomorphic to a part of cyclic group algebra and that any cyclic P-I filter with unit sample response element from this part of cyclic group algebra has the structures of both

cyclic and abelian P-I filters inherently. In this section we undertake a study of cyclic mappings based on another important type of mapping namely CRT. This type of mapping is possible only when the orders of component cyclic sub groups are pairwise relatively prime. In this case it will be shown that algebras are isomorphic. As a consequence of this a cyclic P-I filter can be realized as an equivalent abelian P-I filter. This type of realization has advantages compared to the realization based on G-T algorithm as will be shown later.

We begin by showing that under CRT mapping algebras are isomorphic.

**Theorem 3.7.1:** Let  $n = \prod_{\alpha=0}^{r-1} m_{\alpha}$  be a product of integers relatively prime in pairs. Group algebra  $RG$  of a group  $G$  where  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$  is isomorphic to the group algebra  $RC$  of a cyclic group of order  $n$  under CRT mapping.

To prove this theorem we need the following two lemmas.

**Lemma 3.7.1:** Under CRT mapping, if  $a(X)$  and  $b(X)$  of  $RG$  are mapped into  $a(Z)$  and  $b(Z)$  of  $RC$  then  $a(X) + b(X)$  is mapped into  $a(Z) + b(Z)$ .

**Proof:** Any  $a(X)$  and  $b(X)$  of  $RG$  can be written as  $a(X) = \sum_{i=0}^{n-1} a_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right)$  and

$b(X) = \sum_{i=0}^{n-1} b_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right)$ . Under CRT mapping,

$$a(X) \xrightarrow{\text{CRT}} a(Z) = \sum_{I=0}^{n-1} a_I Z^I \text{ and } b(X) \xrightarrow{\text{CRT}} b(Z) = \sum_{I=0}^{n-1} b_I Z^I$$

where  $I = \sum_{\alpha=0}^{r-1} i_{\alpha} M_{\alpha} N_{\alpha} \pmod{n}$ .

Consider the CRT mapping of  $[a(X) + b(X)]$ ,

$$\begin{aligned} \left[ \sum_{i=0}^{n-1} a_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right) + \sum_{i=0}^{n-1} b_i \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right) \right] &= \sum_{i=0}^{n-1} (a_i + b_i) \left( \prod_{\alpha=0}^{r-1} X_{\alpha}^i \right) \\ &= \xrightarrow{\text{CRT}} \sum_{I=0}^{n-1} (a_I + b_I) Z^I \end{aligned} \quad (3.7.1)$$

Now considering the sum of CRT mapping of  $a(X)$  and  $b(X)$ , we have

$$a(X) + b(X) \xrightarrow{\text{CRT}} \left( \sum_{I=0}^{n-1} a_I Z^I \right) + \left( \sum_{I=0}^{n-1} b_I Z^I \right) = \sum_{I=0}^{n-1} (a_I + b_I) Z^I \quad (3.7.2)$$

From (3.7.1) and (3.7.2), it can be seen that CRT mapping of  $[a(X) + b(X)]$  is same as that of  $a(X)$  and  $b(X)$ .  $\square$

**Lemma 3.7.2:** Let  $G = C_{m_0} \otimes C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$ ; let  $X_\alpha$ ,  $\alpha \in Z_r$ , be a generator of a cyclic sub group  $C_{m_\alpha}$  of order  $m_\alpha$ , let  $Z$  be the generator of a cyclic group  $C_n$  of order  $n$  where  $n = \prod_{\alpha=0}^{r-1} m_\alpha$ . Under a mapping based on CRT  $G$  and  $C_n$  are isomorphic.

**Proof:** Under a mapping based on CRT,  $i_\alpha = I \bmod m_\alpha$  and  $I = \sum_{\alpha=0}^{r-1} i_\alpha M_\alpha N_\alpha \pmod{n}$  where  $M_\alpha = n/m_\alpha$  and  $N_\alpha$  is the solution of  $N_\alpha M_\alpha + n_\alpha m_\alpha = 1$ . Then  $\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha}$  and  $\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}$  are mapped into  $Z^I$  and  $Z^J$  respectively. Consider the product of CRT mappings of  $Z^I$  and  $Z^J$ ,

$$Z^I \xrightarrow{\text{CRT}} \prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha} \text{ and } Z^J \xrightarrow{\text{CRT}} \prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}$$

$$(Z^I Z^J) \xrightarrow{\text{CRT}} \left( \prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha} \cdot \prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha} \right) = \left( \prod_{\alpha=0}^{r-1} X_\alpha^{(i_\alpha + j_\alpha) \bmod m_\alpha} \right) \quad (3.7.3)$$

Now considering the CRT mapping of  $Z^{I+J}$ , we have

$$Z^{I+J} \xrightarrow{\text{CRT}} \left( \prod_{\alpha=0}^{r-1} X_\alpha^{(i_\alpha + j_\alpha) \bmod m_\alpha} \right) \quad (3.7.4)$$

Since (3.7.3) and (3.7.4) are equal  $G$  and  $C_n$  are isomorphic under CRT mapping.  $\square$

**Proof:** In order to prove that  $RG$  and  $RC$  are isomorphic, first we prove that  $[a(X) + b(X)]$  is same as that of  $[a(Z) + b(Z)]$  under CRT mapping, where  $a(X) \xrightarrow{\text{CRT}} a(Z)$  and  $b(X) \xrightarrow{\text{CRT}} b(Z)$ . This is already proved in Lemma 3.7.1. Next, we prove that the product

of any  $a(X)$  and  $b(X)$  of  $RG$  under CRT mapping, is the same as that of  $a(Z)$  and  $b(Z)$  of  $RC_n$ . Since an element of  $RG$  is a linear combination of the elements of  $G$  and linearity property holds for CRT mapping, it is enough to show that the CRT mapping of the product of any two elements of  $C_n$  is the same as that of the product of the CRT mapping of the individual elements of  $C_n$ . This is proved in Lemma 3.7.2.  $\square$

**Corollary 3.7.1:** Under CRT mapping, product of  $a(Z) b(Z) \bmod (Z^n - 1)$  in  $RC$  is the same as that of  $a(X) b(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), \dots, (X_{r-1}^{m_{r-1}} - 1)$  in  $RG$  where  $a(Z) \xrightarrow{\text{CRT}} a(X)$  and  $b(Z) \xrightarrow{\text{CRT}} b(X)$ .

**Proof:** Its proof is contained in the proof of Theorem 3.7.1.  $\square$

**Corollary 3.7.2:** Under CRT mapping, an ideal of  $RC$  mapped into an ideal of  $RG$ .

**Proof:** This a consequence of the above corollary and the Lemma 3.7.1.  $\square$

**Remark 3.7.1:** Theorem 3.7.1 has an interesting consequence that any cyclic P-I filter whose dimension is a product of pairwise relatively prime integers, has an equivalent realization as an abelian P-I filter. As proved in the theorem relevant mapping is given by the CRT. It will be shown later that the realization of abelian P-I filter has some advantages compared to the realization of the corresponding cyclic P-I filter using Good-Thomas algorithm in respect of address shuffling.

We illustrate above concepts with the following example.

**Example 3.7.1:**  $n = 6$  ;  $m_0 = 2$  ,  $m_1 = 3$ .

CRT mapping :  $i = 4 i_1 + 3 i_0 \bmod 6$  where  $i \in Z_6$  ,  $i_1 \in Z_3$  and  $i_0 \in Z_2$ .

Consider the following cyclic P-I filter.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} s_0 & s_5 & s_4 & s_3 & s_2 & s_1 \\ s_1 & s_0 & s_5 & s_4 & s_3 & s_2 \\ s_2 & s_1 & s_0 & s_5 & s_4 & s_3 \\ s_3 & s_2 & s_1 & s_0 & s_5 & s_4 \\ s_4 & s_3 & s_2 & s_1 & s_0 & s_5 \\ s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

We know that the above matrix computation is equivalent to the polynomial multiplication  $a(Z)s(Z) \bmod (Z^n - 1)$ . By Theorem 3.7.1 this is equivalent to the polynomial multiplication  $a(X)s(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), \dots, (X_{r-1}^{m_{r-1}} - 1)$ . The corresponding mapping is given by CRT mapping. The corresponding matrix equivalent is given by

$$\begin{bmatrix} b_0 \\ b_3 \\ b_4 \\ b_1 \\ b_2 \\ b_5 \end{bmatrix} = \begin{bmatrix} s_0 & s_3 & s_2 & s_5 & s_4 & s_1 \\ s_3 & s_0 & s_5 & s_2 & s_1 & s_4 \\ s_4 & s_1 & s_0 & s_3 & s_2 & s_5 \\ s_1 & s_4 & s_3 & s_0 & s_5 & s_2 \\ s_2 & s_5 & s_4 & s_1 & s_0 & s_3 \\ s_5 & s_2 & s_1 & s_4 & s_3 & s_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_3 \\ a_4 \\ a_1 \\ a_2 \\ a_5 \end{bmatrix}$$

Corresponding cyclic P-I filter output is obtained by applying inverse CRT mapping to the output of abelian P-I filter.

Since the extension field required for defining both the DFT and GWHT are same, it is possible to consider the spectral equivalent of the corresponding sample domain CRT mapping. In the following theorem we show that the CRT mapping considered in the sample domain also maps a GWHT coefficient to a corresponding DFT coefficient.

**Theorem 3.7.2:** Let 't' be the CRT mapping; let  $A^D$  and  $A^G$  denote the DFT and GWHT of  $a$ . Then  $A_j^G$  is equal to  $A_{t(j)}^D$ .

**Proof:** GWHT defined with respect to  $\{m_0, m_1, \dots, m_{r-1}\}$  maps a vector  $a$  as

$$A_{\langle j_{r-1} j_{r-2} j_1 j_0 \rangle}^G = \sum_{i=0}^{n-1} a_{\langle i_{r-1} i_{r-2} i_1 i_0 \rangle} [\gamma_{m_{r-1}}^{i_{r-1} j_{r-1}} \gamma_{m_{r-2}}^{i_{r-2} j_{r-2}} \dots \gamma_{m_1}^{i_1 j_1} \gamma_{m_0}^{i_0 j_0}] \quad (3.7.5)$$

where  $\langle i_{r-1} i_{r-2} i_1 i_0 \rangle$  and  $\langle j_{r-1} j_{r-2} j_1 j_0 \rangle$  are the mixed-radix representations of  $i$



and  $j$ .

Under CRT mapping,  $i$  and  $j$  are mapped into  $t(i)$  and  $t(j)$  where  $t(i) = [ \sum_{\alpha=0}^{r-1} N_{\alpha} M_{\alpha} i_{\alpha} ] \bmod n$  and  $t(j) = [ \sum_{\alpha=0}^{r-1} N_{\alpha} M_{\alpha} j_{\alpha} ] \bmod n$  and  $t(i) \bmod m_{\alpha} = i_{\alpha}$ ;  $\alpha \in Z_r$ , and also  $t(i)t(j) = t(ij) \bmod n$ .

Applying CRT mapping to (3.7.5), we get,

$$\sum_{i=0}^{n-1} a_{t(i)} \gamma_n^{t(ij)} = \sum_{i=0}^{n-1} a_{t(i)} \gamma_n^{t(i)t(j)} \quad (3.7.6)$$

which is same as  $A_{t(j)}^D$ .

Therefore  $t(j)^{\text{th}}$  DFT coefficient of 'a' is same as the  $j^{\text{th}}$  GWHT coefficient  $\square$

**Remark 3.7.2:** This is unlike the case of Good-Thomas (G-T) algorithm which requires two different address shuffling, one for the spectral domain and the other for the sample domain. Therefore there is an implementation advantage in using GWHT compared to G-T algorithm in realizing a cyclic P-I filter in that we can do with only one type of address shuffling both in the spectral domain and sample domain, namely CRT mapping.

The block diagrams for realizing cyclic P-I filter using distinct approaches of GWHT and G-T algorithm are given in Figure 3.7.1.

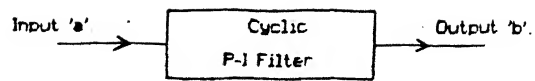


Fig. 3.7.1.a: Cyclic P-I Filter

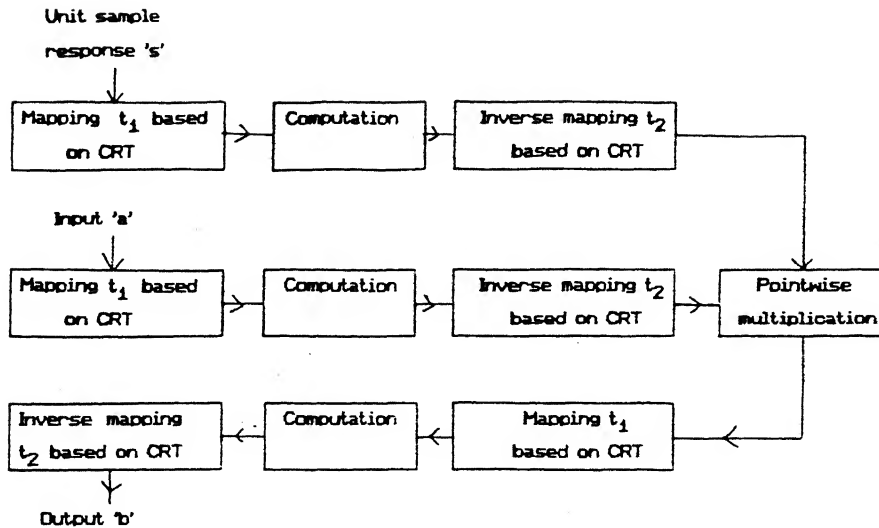


Fig. 3.7.1.b: Implementation of a Cyclic P-I Filter using G-T FFT based on CRT

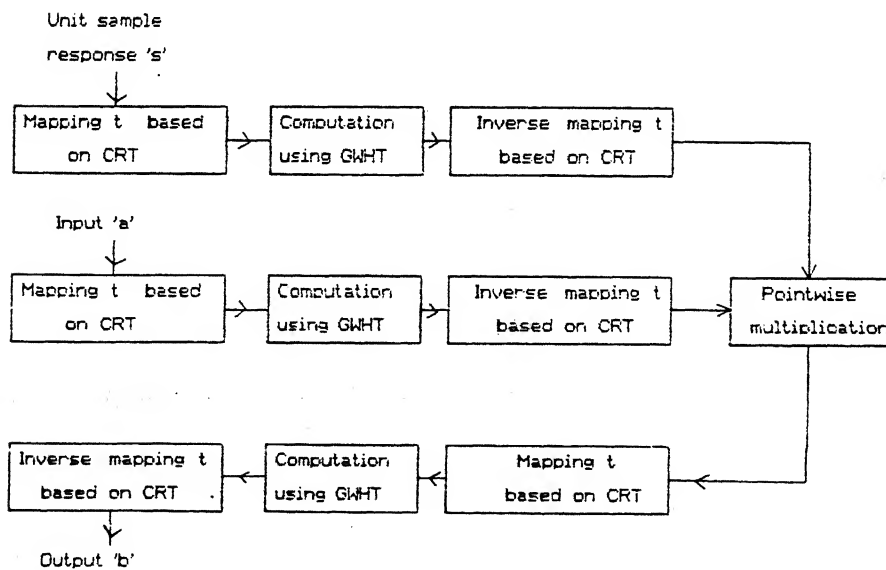


Fig. 3.7.1.c: Implementation of Cyclic P-I Filter using GWHT based on CRT Mapping

Fig 3.7.1 : Implementation of a Cyclic P-I Filter using G-T FFT Algorithm and using GWHT based on CRT

## CHAPTER 4

### CONCLUSIONS

In this thesis, we have considered group algebraic approach to the study of P-I filters. In this approach, a P-I filter is seen as a mapping that maps any element of a group algebra into an element of an ideal in that group algebra. The class of P-I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. This facilitates classification of P-I filters based on ideals in group algebra. Specifically, ideal theoretic results have been used to characterize P-I filters that have got the structures of both cyclic and abelian P-I filters inherently. We call this type of filter as cyclically closed abelian P-I filter. It is shown that under MRX mapping, only a part of abelian group algebra, namely subspace spanned by the class of cyclically closed abelian P-I filters, is isomorphic to the corresponding part of cyclic group algebra. Next, we have considered CRT mapping (When the orders of component cyclic groups are pairwise relatively prime) and shown that under this mapping, a cyclic group algebra is isomorphic to an abelian group algebra. Then we have shown that all cyclic P-I filters whose dimensions are products of pairwise relatively prime numbers have equivalent abelian P-I filter realizations.

#### 4.1 Summary of the Results

The results obtained in this thesis are summarized below.

- 1) A class of P-I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. Consequently, abelian P-I filters are classified based on the ideals in abelian group algebra and cyclic P-I filters are classified based on the ideals in cyclic group algebra.

- 2) A P-I filter that is invariant relative to both cyclic and abelian permutations is

called a cyclically closed abelian P-I filter. This type of P-I filters are interpreted as elements of cyclically closed ideals in abelian group algebras and a complete characterization of these filters is obtained. In the group algebra  $RG$  with  $G = C_{m_0} \otimes$

$C_{m_1} \otimes \dots \otimes C_{m_{r-1}}$  the following elements are cyclically closed abelian filters:

- a) All elements of the ideals of the type  $\langle \theta(x_0) \prod_{k=1}^{r-1} R_0(x_k) \rangle$ .
- b) All elements of the subspace consisting of elements of the form  $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)$  belonging to an ideal of the type  $\langle \theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \rangle$  for  $i = 1$  to  $r-1$ .

c) Elements obtained as linear combinations of the filters of above types.

3) By making use of the 1-D results, a procedure for the identification of 2-D cyclically closed abelian P-I filters is obtained.

4) It is shown that under CRT mapping cyclic and abelian group algebras are isomorphic and that any cyclic P-I filter whose dimension is a product of pair wise relatively prime integers has an equivalent abelian P-I filter realization. An important observation here is that there is an implementation advantage in using GWHT compared to G-T algorithm approach in realizing a cyclic P-I filter.

From the above results, it is concluded that in general GWHT can be used to extract only few cyclic features. These are the features of cyclically closed abelian P-I filters. However, if a GWHT matrix is a direct product of pairwise relatively prime length DFT matrices, then that GWHT can be used to extract all the corresponding cyclic features.

## 4.2 Suggestions for further research

1) In this thesis, we considered characterization of P-I filters that are invariant relative to a cyclic TAP group and a non-cyclic TAP group. Investigation can be done for

characterizing P-I filters that are invariant relative to a pair of non-cyclic TAP groups.

2) Based on the method of characterizing 1-D cyclically closed abelian P-I filters and experimental verifications, a procedure is given for identifying 2-D cyclically closed abelian P-I filters. Therefore, one can work out the theoretical details concerning the characterization of 2-D cyclically closed abelian P-I filters.

3) It is interesting to investigate whether there are any 2-D P-I filters that are invariant relative to two pairs of non-cyclic TAP groups.

4) It remains to be seen whether the number of zero crossings of the basis functions of GWHT can provide a necessary basis for developing the notions of filtering using P-I filters analogous to that of cyclic and dyadic P-I filters.

## REFERENCES

1. Siddiqi. M. U. and V. P. Sinha., 'Signals and Systems over Finite Groups and Monoids', A Short Course on Digital Signal Processing for BEL Engineers, Dept. of Electrical Engineering, IIT Kanpur, 1988.
2. Henning F. Harmuth., 'Transmission of Information by Orthogonal Functions', Springer-Verlag, New York, 1969.
3. Siddiqi. M. U., 'A Study of Permutation-Invariant Linear Systems', Ph.D. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1976.
4. V. P. Sinha., 'Signal Spaces for Digital Filters', A Short Course on Digital Signal Processing for BEL Engineers, Dept. of Electrical Engineering, IIT Kanpur, 1988.
5. Huffman and Kunze., 'Linear Algebra', Prentice Hall of India, New Delhi, 1986.
6. Sundar Rajan. B., 'Spectral Characterization of Abelian group codes', M.Tech. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1984.
7. Madhusudhana. H. S., 'On Abelian codes which are closed under Cyclic shifts', M.Tech. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1987.
8. T. Beth., 'Generalizing the Discrete Fourier Transform', Discrete Math., vol.56, no. 2-3, pp 95-101, 1985.
9. P.R.K. Rao., 'Certain Generalizations of Permutation-Invariant Systems', Ph.D. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1978.